THE DERIVE - NEWSLETTER #108

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THE BULLETIN OF THE DERIVE USER GROUP

+ CAS-TI

Contents:

December 17/January 2018

For all of you who understand German:
Thomas Müller (PH Krems, NÖ) informiert über aktuelle Tätigkeiten von T^3 Österreich:

Seit kurzem gibt es nun neben den NSPIRE-Unterrichtsmaterialien für die 5. und 6. Klasse (AHS) auch zu den Lehrplanthemen der 7. Klasse (11. Schulstufe) konkrete Unterrichtsmaterialien zum Technologieeinsatz. Diese umfassen die Bereiche Differentialrechnung, Gleichungen und Zahlenmengen, Kurven und Flächen sowie Stochastik, alles abrufbar unter

http://www.t3oesterreich.at/index.php?id=215

Eno Tõnisson from Tartu, Estonia, sent great news: he finished his PhD and invites to download his PhD paper:

Differences between Expected Answers and the Answers Offered by Computer Algebra Systems to School Mathematics Equations

Download from: http://dspace.ut.ee/handle/10062/58398

Here is a great collection of LUA-programs for TI-Nspire:
http://www.ticalc.org/pub/nspire/lua/math/

A rich collection of interesting links is available:
http://condellpark.com/kd/Robinson_VisualMathematics.pdf

Browse a very rich in content website in French (Jean Michel Ferrard):
http://www.mathprepa.fr/

The next links are provided by Prof. Michael de Villiers in his latest Math-e Newsletter:

University lecturers or lecturers of advanced calculus at school, might enjoy read the interesting, informative paper by David Bressoud: "Historical Reflections on Teaching the Fundamental Theorem of Integral Calculus".

Watch the popular public lecture "How I became interested in the foundations of mathematics" by acclaimed mathematician Vladimir Voevodsky at the Asian Science Camp 2015.

Read MAA Convergence: Mathematics History for Your Classroom for many useful ideas on how to use the history of mathematics in the classroom.

The new Standards for Preparing PreK-12 Teachers of Mathematics adopted by the Association of Mathematics Teacher Educators (AMTE) are available free online at Teacher Preparation Standards.

Incorporate real world mathematical and physical modelling by video-taping and analyzing real situations in your classroom by downloading the free Tracker software.

Read for free "Recent research on geometry education: an ICME-13 survey team report" in ZDM, 2016, 48:691-719.

Watch Paul Ernest's talk "Mathematics, Beauty and Art: What is beauty in mathematics?" at the MACAS 7 conference in Copenhagen on Tuesday 27 June, 2017.

Download the open access Proceedings of the 13th International Congress on Mathematical Education ICME-13, Editor, Gabriele Kaiser.

Dear DUG Members,

As I announced in my last letter this newsletter is very late. This is due to two reasons: the first one is our very extended travel in December/January to Myanmar. We had a wonderful magic time in former Burma. Weather was great, we had a very harmonic travel group, an excellent female tour guide and met a - at least for us – unknown world.

The second reason is that the two main contributions of this DNL were very interesting but very laborious as well. They needed a lot of communication between the authors and me and finally some other DUG members were involved, too. Many thanks to you all for your wonderful and valuable communication and support. Special thank goes to Rolf Pütter who extended his LUA-TI Reuleaux program for Wolfgang Alvermann's contribution.

David Halprin sent three or four versions of his "Glockenspiel" until he was really satisfied with his paper. What if this had been in times of surface mail only?

I mentioned Rolf Pütter. I'd like to take the occasion to invite you to take notice of the many links given on the information page: Thomas Müller offers for all of you who understand German a lot of new materials produced by members of T^3-Austria. It is a pleasure for me to distribute the link to Eno's PhD work. Eno was one of the organizers of TIME 2012 in Tartu, Estonia.

If you are interested in LUA programming then the LUA-website should be a "MUST". Have a look and be amazed.

The last two pages of this issue are dedicated to two innovative TI-products: the TI-Innovator™ Hub and the Rover. We plan to publish applications for these two devices in future DNLs. I know that a group of teachers (Austria and Germany) are working on a collection of experiments which can be performed in class room. I had the occasion to run the Rover on my floor and table. Links to more information and introductory courses are provided.

Best regards and wishes

Josef

The Shwedagon Pagoda in Yangon and a picture from the balloon gliding above Bagan

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2018

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What´s new? J. Böhm, AUT
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZK
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
A New Approach to Taylor Series, D. Oertel, GER
Rational Hooks, J. Lechner, AUT
Statistics of Shuffling Cards, H. Ludwig, GER
Charge in a Magnetic Field, H. Ludwig, GER
Factoring Trinomials, D. McDougall, CAN
Selected Lectures from TIME 2016
Programming the TI-Innovator™ Hub and TI-Innovator™ Rover
A Temperature Warner realized with TI-Innovator, H. Langlotz a.o., GER
and others

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Glockenspiel (*Carillon*)

David Halprin, Australia

Herr Magister Josef Böhm

Wie geht es ihnen? Hoffentlich, du bist sehr gesund, deine Frau auch.

Entschuldigen sie sich mich, weil nun, muss ich in English schreiben, henceforth.

Methought `twas better to get straight into it.

Author's Challenge

Does analysis verify intrinsic detail?

Consequential Intention of this Paper

How a lay person reconfirms integrated notations.

A logical analysis of the GLOCKENSPIEL type of problem & its ilk

When I read DNL107 last week, it was just before bedtime and so I had the geometric sequence from Peter going thru my mind. I decided that I would **NOT** look up any of your, (or his), citations until I grappled with it in my own way, and therefore not be influenced into an extraneous way of thinking.

I strongly suspected something about it, that had not been mentioned in the DNL, and voilà, next morning I confirmed my suspicions.

I decided, there and then, that it deserved a name, so I have named it "*Glockenspiel-2*" in deference to Danny Kaye and his rendition of "*Tubby the Tuba*", (written by his wife Cynthia Fine), and in which he made a startling 'discovery', that the Tuba was a "*Glockenspiel in Disguise*". (You can find it on YouTube, it's great entertainment.)

Many years ago, I had a similar experience of finding a Quattracci series, (recursive over four terms), in disguise, when researching the geometrical investigative papers from 1847 of Charles Dupin, wherein he showed a unique method for interpolating missing data from a statistical table with incredible accuracy, far beyond traditional methods. So, I have written three separate papers DUPINACCI-4.pdf, DUPINACCI-6.pdf and DUPINACCI-8.pdf, yet to be published.

My first approach, which led me nowhere significant was simply: -

$$z+1=x^2 \qquad \frac{z}{2}+1=y^2 \quad \therefore \quad 2y^2-x^2-1=0$$

This is a bivariate quadratic, for which only integer solutions must obtain. However, you may remember sometime back in the dark ages, that one of your members wrote a paper on recursive series, and gave a simple proof, that all recursive series have the property, that the ratio of two neighbouring terms asymptotes to a limit as n approaches infinity[1]. This can be expressed as a syllogism in any language of binary logic:

(Binary logic admits each statement to be either true or false, and is therefore most appropriate for mathematics and theoretical physics, since a ternary logic, (or higher logic), subsumes probability and therefore uncertainty leading to unacceptable conclusions, with resultant lack of rigour in those fields, hence this paper strives for such sufficient *Genauigkeit* as would satisfy the reader(s).)

[1] David Halprin, in DNL93, Recursive Series of Numbers, An Umbral Look

Apart from Boolean Algebra and the currently espoused symbolic language symbols, there is a far superior symbology, originated by a Polish academic, Jan Lukasiewicz (1878-1956) in 1924.

Professor Layman E. Allen from Yale, and later the University of Michigan, Ann Arbor), published the game *Wff'n Proof* in 1962, which is much more than a game of skill, in that it teaches every competitor symbolic logic, using Lukasiewicz notation even if the contestants hadn't shown a prior interest in it.

Wff'n Proof is a shortened form of "*Well-Formed Formula Expression and Proof*".

In fact, there have been inter-school and inter-varsity competitions going on for over five decades. When one reads the manual or views Professor Serna on YouTube demonstrating it, it proves to be extremely intriguing and rewarding.

1) When Hewlett Packard introduced its hand-held calculators HP-25, HP-35, HP-41C etc., they used the mirror image of this notation, calling it "Reverse Polish Notation". HP used the arithmetical operators **+, –, x,** ÷ as postfix operands.

2) Standard logic symbology →, ↔, ¬, ∩, ∪, is used, as in standard algebra, by employing the assorted connectives as is done in written prose, and which are called infix operands and thereby it necessitates the use of brackets.

3) Polish Notation, (Lukasiewicz), uses the letters **N**, **A**, **C**, **E**, **K** as prefix operands and thereby results in parenthesis-free expressions, no matter how long the chain of letters may be to represent a compound statement.

There are seemingly hundreds of connectives for joining two statements in most European languages, especially the English language. Despite this proliferation and their individual nuances of meaning, when analysed logically, they reduce to ten conjunctions. The truth table, that subsumes them has 16 columns, since it also includes a tautology, a contradiction, two affirmations and two denials:

TRUTH TABLE

				POWER	OR					XNOR	AND	NAND	XOR	NOT		NOT		NOR	GROUND
p	q	Np	Nq		Apq	Cqp	p	Cpq	q	Epq	Kpq	NKpq	NEpg	Nq	KpNq	Np	KNpq	NApq	
				1	2	3	4	5	6	7	8	-8	-7	-6	-5	-4	-3	-2	-1
t	**t**	**f**	**f**	t	t	t	t	t	t	t	t	f	f	f	f	f	f	f	f
t	**f**	**f**	**t**	t	t	t	t	f	f	f	f	t	t	t	t	f	f	f	f
f	**t**	**t**	**f**	t	t	f	f	t	t	f	f	t	t	f	f	t	t	f	f
f	**f**	**t**	**t**	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f

Explanation of the columns 1 through 8 and -8 through -1:

Col 1: Tautology: always true
Col 2: Inclusive Disjunctive (OR)
Col 3 & Col 5: Conditional, Inclusion & sometimes Implication
Col 4 & Col 6: Affirmation
Col 7: XNOR, Biconditional, Equivalence
Col 8: Conjunction (AND)

Col -8: Non-Conjunction (NAND)
Col -7: Exclusive Disjunctive, Non-Equivalence (XOR)
Col -6 & Col -4: Denial (NOT)
Col -5: Non-Implication
Col -3: Non-Inclusion
Col -2: Non-Alternation (NOR), Joint Denial
Col -1: Contradiction: always false

e.g.1) **Col -8:** The Sheffer Stroke is the **NAND** operator. (Non-Conjunction)

NKpq = ApNq
Not (p and q) = Either p or not q

e.g.2) **Col 1:** The Law of Non-Contradiction (De Morgan Rule)

NKpNp = ApNp
Not(p and Not p) = Either p or Not p

e.g.3) **Col 1:** The Law of Excluded Middle (De Morgan Rule)

ApNp = NKNpp
Either p or Not p = Not(p and Not p)

This demonstrates the equivalence of examples 2 & 3 despite their different wording in their text definitions, thus showing the De Morgan Laws.

e.g.4) **Col 7: XNOR (Biconditional)**

Epq = NKApqANpNq = NKApqNKpq = NAKpNqKNpq = KCpqCqp

Equivalent English expressions for e.g.4):

Epq	p if and only if q
NKApqANpNq	It is not the case that “(p or q) and (not p or not q)”
NKApqNKpq	It is not the case that “Either p or q, but not both”
NAKpNqKNpq	It is not the case that “(p but not q) or (q but not p)”
KCpqCqp	(If p then q) and (If q then p)

viz. “*I shall solve my series problem if I adopt the correct analytical approach*” and “*I shall adopt the correct analytical approach if I solve my series problem*”.

e.g.5) **Col 2: OR (Inclusive Disjunctive)**

a) Apq; p and/or q; either p or q or both p and q; p vel q
This is also called *Inclusive Disjunction* and *Alternation*.

b) NKNpNq Not “Neither p nor q”
It is not the case that “Neither p nor q”

c) Contrapositive pair, each of which represents **OR**
CNpq If not p then q
CNqp If not q then p

e.g.6) **Col -7: XOR (Exclusive Disjunctive, Non-Equivalence)**
The exclusive disjunctive has four equivalent forms in terms of N, E as well as other forms in terms of A, K, N ; C, K, N.

1a) NEpq ; NEqp
- i) Not "p iff q"
- ii) Not "p unless not q"
- iii) Not "p provided that q"
- iv) Not "p is both necessary and sufficient for q"
- v) Not "q is both necessary and sufficient for p"
- vi) Not "q iff p", Not “q if and only if p"

1b) ENpq
- i) Not p provided that q
- ii) Not p with the proviso that q
- iii) Not p as long as q
- iv) Not p unless not q
- v) Not p iff q
- vi) q unless p
- vii) Not "Either q or not p"
- EqNp q iff not p

1c) EpNq
- i) p unless q
- ii) p iff not q
  p provided that not q
- iii) ENqp Not q iff p

1d) NENpNq
- i) Not "Not p iff not q"; Not "Not q iff not p"
- ii) Not "Not p unless q"; Not "not q unless p"
- iii) “Either not p or not q” but not both “not p and not q”
- iv) “Either not p or not q” and “Either p or q”; “Either not p or not q” and “p or q”

NENqNp
  Not "Not q unless not p"

2a) KApqANpNq
  p or q; not p or not q

2b) KApqNKpq
  p or q ; q or p ; p aut q
  Either p or q but not both

3) AKpNqKNpq
  p but not q ; q but not p

4) NKCpqCqp
  It is not the case that "(q if p) and (p if q)"

Syllogisms are part and parcel of the game. Especially eye-opening to the uninitiated is the equivalence of some compounded statements and their ability to be converted from one connective to another via the De Morgan Laws and/or other comparable laws

To illustrate the inter-conversions between connectives:

DE MORGAN’S LAW

This shows one how to relate the “and/or” connective to “and”. Firstly, to state it in words, then in symbols.

In order to change a compounded statement from the one connective to the other, without changing the truth value, firstly negate each component and then negate the whole compound statement.

Kpq becomes NANpNq and similarly Apq becomes NKNpNq hence there are four examples of each conversion, since each statement may or may not be negated. In both cases, the order of appearance of the component statements is irrelevant.

CONTRAPOSITIVE RULE

To find the contrapositive form of a compound statement using the “if … then” connective, one negates each statement and reverses the order, hence:

Cpq → CNqNp

Similarly, there are four examples of such transposition.

LAZER’S LAWS

1a) To find the relationship changes required, when converting from “and/or” to “if … then”, all one does is negate either one of the component statements and put this one before the other.

Apq becomes CNpq or the contrapositive form CNqp.

1b) To convert back from “if … then” to “and/or” all one does is negate the first component statement and interchange connectives, the order, then, of the components being irrelevant.

Cpq → ANpq (AqNp)

2a) To find the relationship changes required when converting from “and” to “if … then” the rule is: Negate one of the components, place it second, then negate the whole.

Kpq becomes NCpNq or its contrapositive NCqNp.

2b) To go back from”If – then” to “and”, all one does is negate the second component statement, interchange connectives and negate the whole, (the order of appearance is irrelevant).

Cpq → NKpNq

EQUIVALENCE RULES

There are four equivalent forms for any compounded statement using the connective of equivalence.

Starting with one, the second is found by negating each of its component statements. The third may be obtained from the first by negating only one of the component statements and then negating the whole compound statement. The fourth is obtained from the third as the second was from the first, hence:

1a)	Epq
1b)	NENpq
1c)	NEpNq
1d)	ENpNq
2a)	NKApqANpNq
2b)	NKApqNKpq
3)	NAKpNqKNpq
4)	KCpqCqp

Also, in mathematics, especially, one must be aware of *"The Principle of Sufficient Reason"*:

If p is true, then there is a sufficient explanation/proof of why this is true.

(It is a great tragedy that most theoretical physicists of today do NOT observe this principle.)

e.g.1

General Syllogism for Series	**Lukasiewicz Notation**	
Major Premise: The series is recursive	q	
Minor Premise: $T_n/T_{n-1} \to$ a limit as $n \to \infty$	p	
Initial Conclusion	p and q	Kpq
Final Conclusion	p iff q	Epq

This conclusion is easily explained, because both premises are known to be incontrovertibly true.

e.g.2

Glockenspiel-2 Syllogism	**Lukasiewicz Notation**
MajorPremise: $T_n/T_{n-1} \to$ a limit as $n \to \infty$	p
Minor Premise: The series is recursive	q
Conclusion If p then q	Cpq

Explanation

We already know from our study of recursive series:

"if q then p", (Cqp), so as a result of the finding that Glockenspiel-2 is recursive, then it follows that we have both Cpq and Cqp, which is equivalent to saying "p if and only if q" [(p iff q) or Epq].

KCpqCqp = Epq (They have the same truth table)

This is justified because we have now proven that Glockenspiel-2 is recursive, so both premises are true, as in e.g.1 above.

To quote a macaronic, (based on his Gedanken Experiment), often stated by my philosophy mentor, Herr Doktor Beissendkopf Zweistein:-

> *"Wenn es spaziert wie ein Duck, es sieht aus wie ein Duck, es quakt wie ein Duck, dann vielleicht es ist ein Duck."*

e.g.3

Syllogism for an Unknown Series	**Lukasiewicz Notation**	
Major Premise: A given defining property	p	
Minor Premise: The series is recursive	q	
Conclusion	If p then q	Cpq

If we start with a given property, that defines the series, and in addition we can prove that the series is recursive, the Cpq obtains, but we cannot assume Epq until such time that the property in the Major Premise is a property of **ALL** recursive series.

For many years, I subscribed to the Fibonacci Quarterly, and eventually relinquished my membership due to other interests in mathematics.

One thing that stood out from the many papers therein, was that there seemed to be no end to the number of properties of all recursive series. This illustrated that, starting with Recursive Series, one obtains assorted alternate definitions for any such series.

But if one is given such a property of an unknown series, one has to check it out to see if is has its "secret" recursive identity.

Hence:

p = a property of an unknown series
q = the series is recursive

therefore, if we can identify that it is recursive, then we have Cpq as in the case of Glockenspiel-2, and then we also have Cqp, giving us the final conclusion for Glockenspiel-2 to be Epq.

So, when I saw that problem, I suspected that it was probably recursive, and that it was incumbent upon me to prove it for this special case, even though one cannot extrapolate to other cases, unfortunately.

I quickly eliminated it being recursive over two terms, so I needed to test it for Tribonacci type sequences.

I just used the 8 terms by taking them in six groups of three terms

$$T_0 - T_2,\ T_1 - T_3,\ T_2 - T_4,\ T_3 - T_5,\ T_4 - T_6,\ T_5 - T_7$$

and substituting them into the expression $T_n = p \cdot T_{n-3} + q \cdot T_{n-2} + r \cdot T_{n-1}$ consistently resulting in $p = 1$, $q = -35$ and $r = 35$.

[You may remember, that it was illustrated in that DNL recursive paper, that recursive series are surreptitious, ("in disguise"), in the natural number system, since those numbers can be shown to be both a Fibonacci-type sequence (m=2) and a Tribonacci-type sequence (m=3). In fact, I believe that the number of recursive terms for the natural numbers can be for any integer value of m.]

So, having established that Peter's sequence is "secretly" a recursive series, it deserves the name *Glockenspiel-2*.

One of the beauties of mathematics, is that once you have investigated a particular family and therefore found many of its properties, then at a later time you find that a new investigation reveals a kinship, (shared DNA), with said family, then you know immediately, that it must have the same structure, (genetic make-up), and can be welcomed as a full "sibling".

Now, as you may remember, there are many ways to define any recursive series:

e.g.
1) Each term is the sum of the previous m terms (m = 2, 3, 4, etc.)
(Glockenspiel-2: $p = 1, q = -35, r = 35$)
2) The generating function. (see below)
3) An equation, which determines the n^{th} term (see below)
4) There is limiting value of the ratio of two successive terms (see below)
5) The sum to the n^{th} term (see below)
6) There is limiting value of the ratio of $S_n./T_n$ (see below)

I have shown the first four in my Derive file, (Glockenspiel.dfw), but the stumbling block for me, for finding a sum, is predicated on an expression, whose denominator is (p+q+r-1) and which must not be zero. Since zero denominator is the case with Glockenspiel-2, I have to admit temporary defeat. However, I solved it by another method.

An unexpected discovery by me, at least, perhaps a new discovery for all, or most, readers, is the sixth definition, above.

The Generating Function and the Term T_n

$$S_x = \frac{T_0 + (T_1 - r.T_0).x - (T_2 + r.T_1 + q.T_0).x^2}{1 - r.x - q.x^2 - p.x^3} \qquad \textbf{Generating Function}$$

$$T_n = \frac{a^n}{(a-b)(c-a)}.[T_2 + T_1(b+c) - T_0 bc]$$

$$+ \frac{b^n}{(a-b)(b-c)}.[T_2 + T_1(c+a) - T_0 ca]$$

$$+ \frac{c^n}{(b-c)(c-a)}.[T_2 + T_1(a+b) - T_0 ab]$$

$$\therefore \; S_x = \frac{-48x(70x-1)}{x^3 - 35x^2 + 35x - 1}$$

$$= \frac{\frac{2449\sqrt{2}}{2}}{x - 12\sqrt{2} - 17} + \frac{\frac{6927}{4} - \frac{2449\sqrt{2}}{2}}{x + 12\sqrt{2} - 17} - \frac{207}{2(x-1)}$$

$$T_n = (\sqrt{2}-1)^{4(n-1)}\left(\frac{99}{4} - \frac{35\sqrt{2}}{2}\right) + (\sqrt{2}+1)^{4(n-1)}\left(\frac{99}{4} + \frac{35\sqrt{2}}{2}\right) - \frac{3}{2} \qquad \textbf{The n}^{\textbf{th}} \textbf{ Term}$$

$$= (\sqrt{2}-1)^{4n}.\left(\frac{3}{4} - \frac{17\sqrt{2}}{32}\right) + (\sqrt{2}+1)^{4n}.\left(\frac{3}{4} + \frac{17\sqrt{2}}{4}\right) + \frac{3[n.(2\sqrt{2}-3).(2\sqrt{2}+3)-1]}{2}$$

$$= (\sqrt{2}-1)^{4n}.\left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right) + (\sqrt{2}+1)^{4n}.\left(\frac{3}{4} + \frac{\sqrt{2}}{2}\right) - \frac{3}{2}$$

(see Derive file Glockenspiel.dfw)

My computer was able to factorise up to T_{70}, but T_{100} gave it a "migraine headache" by over-heating its neurons, thus making them jump over their synapses.

$$T_{10} = 2{,}982{,}076{,}586{,}042{,}448 = 2^4.3.19.23.29.41.59.353.5741$$

$$= 10^{15} \times 2.982076586$$

$$T_{15} = 134{,}906{,}383{,}349{,}641{,}674{,}163{,}600$$

$$= 2^7.3.5^2.7.17.29.31^2.41.269.577.665857 = 10^{23} \times 1.349063833$$

$$T_{20} = 6{,}103{,}039{,}859{,}426{,}804{,}588{,}442{,}079{,}582{,}560$$

$$= 2^5.3.5.7^2.13^2.17.19.29.41.59.239.241.4663.5521.45697$$

$$= 10^{30} \times 6.103039859$$

$$T_{25} = 2^4.3.29.41.79.599.1549.22307.29201.33461.66923.45245801$$

$$= 10^{38}.2.760958718$$

$T_{25} = 2^4.3.29.41.79.599.1549.22307.29201.33461.66923.45245801$

$= 10^{38}.2.760958718$

$T_{30} = 2^4.3^2.5^2.7.11.19.29.31^2.41.59.269.601.1301.2281.424577$

$.865097.3272609$

$= 10^{46} \times 1.249032157$

$T_{40} = 2^6.3.17.19.29.41.59.241.577.2297.5521.188801.302663.3553471$

$.9393281.1746860020068409$

$= 10^{61} \times 2.556236509$

$T_{50} = 10^{76} \times 5.231526709$

$T_{60} = 10^{92} \times 1.070670558$

$T_{70} = 10^{107} \times 2.191206330 > 10^7 \times \text{Googol}$

$T_{100} = 10^{153} \times 1.878302404 > 10^{53} \times \text{Googol}$

The Ratio of two Consecutive Terms

Glockenspiel-2 has 3 different roots in the denominator of the generating function (a,b,c).

With the more complicated ratios in Tribonacci-type series, we choose whichever is the largest of the roots and hence find that the limiting value is the value of that largest root.
Take the Glockenspiel case where b < a < c, our ratio is of the form:

$$R_{TribType} = \lim_{n\to\infty} \frac{K.a^n + L.b^n + M.c^n}{K.a^{n-1} + L.b^{n-1} + M.c^{n-1}}$$

$$= \lim_{n\to\infty} \frac{a\left(1 + \frac{L.P^n}{K} + \frac{M.Q^n}{K}\right)}{1 + \frac{L.P^{n-1}}{K} + \frac{M.Q^{n-1}}{K}} = a$$

Glockenspiel values

$a = 1 \quad , \quad b = 17 - 12\sqrt{2} \quad , \quad c = 17 + 12\sqrt{2}$

Where the limiting ratio for Glockenspiel is $17 + 12\sqrt{2}$

$r = a + b + c = 35 \quad , \quad q = -(ab + bc + ca) = -35 \quad , \quad p = abc = 1$

[Comparably, when two roots are equal (a,b,b) where a > b then the limit is a, but if a < b then the limit is b.
When all roots are equal (a,a,a), then a is the limiting value.

When there is only one real root, (a, b ± *i* c)

$$b \pm ic = \sqrt{b^2 + c^2}.\left(\frac{b}{k} \pm \frac{ic}{k}\right) = k.cis(\pm\theta) = k.e^{\pm i.\theta} = k.(\cos\theta \pm i.\sin\theta)$$

$$\text{where} \quad k = \sqrt{b^2 + c^2} \quad , \quad \theta = \arctan\frac{c}{b} \quad , \quad \cos\theta = \frac{b}{k} \quad , \quad \sin\theta = \frac{c}{k}$$

then the limiting ratio is the larger out of a and k]

The Sum of a Number Sequence S_n
Method 1

The sum to n terms for such a series may be deduced:

Let S_n represent the sum to n terms for Tribonacci series:

$$r.S_n = r.T_0 + r.T_1 + r.T_2 + \ldots\ldots + r.T_n$$

$$q.S_n = \quad + q.T_0 + q.T_2 + \ldots\ldots + q.T_{n-1} + q.T_n$$

$$p.S_n = \quad p.T_0 + p.T_1 + \ldots\ldots + p.T_{n-2} + p.T_{n-1} + p.T_n$$

$$(p+q+r).S_n$$

$$= (q+r).T_0 + r.T_1 + (S_n - T_0 - T_1 - T_2) + T_{n+1} + (p+q).T_n + p.T_{n-1}$$

$$S_n = \frac{(q+r-1).T_0 + (r-1)T_1 - T_2 + (p+q)T_n + p.T_{n-1} + T_{n+1}}{p+q+r-1}$$

$$\text{Iff} \quad p + q + r \neq 1$$

This equation simplifies for particular examples e.g. Standard Tribonacci

$$S_n = ½(T_0 - T_2 + T_n + T_{n+2}) = ½(T_n + T_{n+2} - 1)$$

Method 2

The sum to the n^{th} term S_n CANNOT be obtained via Derive's Calculus Sum command. The required sum is subject to strict conditions re the reciprocals of the 3 roots of the denominator of the generating function.

There are 4 basic types, subsuming 9 cases in all, of which 5 involve a root-value of 1.

In the 4 cases where no root is unity, then the 3 partial fractions of T_n have a factor, that is a^n, b^n or c^n, which on summation yields a geometrical progression, the sum of which would have a zero denominator if the reciprocal root = 1. So, in those 5 cases the potential geometric progression is converted to an arithmetic progression, the sum of which is simply n+1.

Hence:

CASE	RECIPROCAL ROOTS	DETAILS
1a: 3 Real Roots	a, b, c	All different $\neq 1$
1b: 3 Real Roots	1, b, c	All different a=1, $b \neq c \neq 1$
2a: 3 Real Roots	a, a, b	$a \neq b \neq 1$
2b: 3 Real Roots	1, 1, b	a = 1, $b \neq 1$
2c: 3 Real Roots	a, a, 1	$a \neq 1$, b = 1
3a: 3 Real Roots	a, a, a	$a \neq 1$
3b: 3 Real Roots	1, 1, 1	a = 1
4a: 1 Real Root	a, $b \pm i\,c$	$a \neq 1$
4b: 1 Real Root	1, $b \pm i\,c$	a = 1

N.B. The Glockenspiel-2 series is in Case 1b.

Remember that the denominator was factorized into three linear factors:

$$r \cdot x - q \cdot x^2 - p \cdot x^3 = (1 - a \cdot x)(1 - b \cdot x)(1 - c \cdot x)$$

Where a, b, and c are the reciprocals of the roots of the cubic

R = a + b + c, -q = ab + bc + ca, p = abc

CASE	p	q	r
1a	abc	-(ab + bc + ca)	a + b + c
1b	bc	-(b + bc + c)	1 + b + c
2a	a^2b	$-(2ab+a^2)$	2a + b
b	b	-(2b + 1)	2 + b
2c	a^2	$-(2a + a^2)$	2a + 1
3a	a^3	$-3a^2$	3a
3b	1	-3	3
4a	$a(b^2 + c^2)$	$-(2ab + b^2 + c^2)$	a + 2b
4b	$b^2 + c^2$	$-(2b + b^2 + c^2)$	1 + 2b

The Ratio $\frac{S_n}{T_n}$

(T_n and S_n are derived in file Glockenspiel.dfw which is following, Josef)

$$\#1:\quad T(n) := (\sqrt{2} - 1)^{4\cdot(n-1)}\cdot\left(\frac{99}{4} - \frac{35\cdot\sqrt{2}}{2}\right) + (\sqrt{2} + 1)^{4\cdot(n-1)}\cdot\left(\frac{35\cdot\sqrt{2}}{2} + \frac{99}{4}\right) - \frac{3}{2}$$

$$\#2:\quad S(n) := (\sqrt{2} - 1)^{4\cdot n}\cdot\left(\frac{3}{4} - \frac{17\cdot\sqrt{2}}{32}\right) + (\sqrt{2} + 1)^{4\cdot n}\cdot\left(\frac{17\cdot\sqrt{2}}{32} + \frac{3}{4}\right) - \frac{3\cdot(n+1)}{2}$$

$$\#3:\quad \frac{S(100)}{T(100)} = 1.030330085$$

$$\#4:\quad \frac{S(200)}{T(200)} = 1.030330085$$

$$\#5:\quad rat := \frac{S(n)}{T(n)}$$

$$\#6:\quad rat := -\frac{\sqrt{2}\cdot((\sqrt{2} - 1)^{4\cdot n}\cdot(12\cdot\sqrt{2} - 17) + (\sqrt{2} + 1)^{4\cdot n}\cdot(12\cdot\sqrt{2} + 17) - 24\cdot\sqrt{2}\cdot(n + 1))}{8\cdot((\sqrt{2} - 1)^{4\cdot n}\cdot(2\cdot\sqrt{2} - 3) - (\sqrt{2} + 1)^{4\cdot n}\cdot(2\cdot\sqrt{2} + 3) + 6)}$$

```
#7:  VECTOR(rat, n, 10, 50, 10)
#8:  [1.030330085, 1.030330085, 1.030330085, 1.030330085, 1.030330085]
```

I try to find the limit of the quotient, multiplying numerator and denominator by $(\sqrt{2} + 1)^{4n}$ but I don't have success.

$$\#9:\quad \frac{\text{EXPAND}(\text{NUMERATOR}(rat)\cdot(\sqrt{2} + 1)^{4\cdot n})}{\text{EXPAND}(\text{DENOMINATOR}(rat)\cdot(\sqrt{2} + 1)^{4\cdot n})}$$

$$\#10:\quad \frac{\sqrt{2}\cdot((\sqrt{2} + 1)^{8\cdot n}\cdot(12\cdot\sqrt{2} + 17) - 24\cdot\sqrt{2}\cdot(\sqrt{2} + 1)^{4\cdot n}\cdot(n + 1) + 12\cdot\sqrt{2} - 17)}{8\cdot((\sqrt{2} + 1)^{8\cdot n}\cdot(2\cdot\sqrt{2} + 3) - 6\cdot(\sqrt{2} + 1)^{4\cdot n} - 2\cdot\sqrt{2} + 3)}$$

$$\#11:\quad \lim_{n\to\infty} \frac{\sqrt{2}\cdot((\sqrt{2} + 1)^{8\cdot n}\cdot(12\cdot\sqrt{2} + 17) - 24\cdot\sqrt{2}\cdot(\sqrt{2} + 1)^{4\cdot n}\cdot(n + 1) + 12\cdot\sqrt{2} - 17)}{8\cdot((\sqrt{2} + 1)^{8\cdot n}\cdot(2\cdot\sqrt{2} + 3) - 6\cdot(\sqrt{2} + 1)^{4\cdot n} - 2\cdot\sqrt{2} + 3)}$$

$$\#12:\quad \frac{\sqrt{2}\cdot(\infty + 12\cdot\sqrt{2} - 17)}{8\cdot(\infty - 2\cdot\sqrt{2} + 3)}$$

Next try is applying the rule of l'Hopital:

$$\#13:\quad \lim_{n\to\infty} \frac{\text{EXPAND}\left(\left(\frac{d}{dn}\ \text{NUMERATOR}(rat)\right)\cdot(\sqrt{2} + 1)^{4\cdot n}\right)}{\text{EXPAND}\left(\left(\frac{d}{dn}\ \text{DENOMINATOR}(rat)\right)\cdot(\sqrt{2} + 1)^{4\cdot n}\right)} = \frac{3\cdot\sqrt{2}}{8} + \frac{1}{2}$$

Now it works!!

The ratio, for Glockenspiel-2, that came out of this analysis was totally unexpected, and which to date, I find inexplicable. However, it simplifies beautifully to exactly

$$\frac{1}{2}+\frac{3}{8}\sqrt{2}.$$

I issue a challenge for the reader to find a mathematical explanation for why it is so. Is it perhaps, a further way of defining all recursive series?

David Halprin, davrin999@gmail.com, 16 December 2017

```
Glockenspiel.dfw  David Halprin   davrin999@gmail.com    November 2017

#1:   [CaseMode := Sensitive, InputMode := Word]

      T_n=p*T_(n-2)+q*T_(n-1) → T_2=p*T_0+q*T_1

#2:   z = p·x + q·y

#3:   SOLVE(1680 = 48·q, q) = (q = 35)

#4:   SOLVE(57120 = p·48 + 35·1680, p) = (p = -35)

      p = -35, hence:

#5:   - 35·1680 + 35·57120 = 1940400

      which is not T_4 = 1940448. This has demonstrated that
      the series is NOT recursive over two previous terms!

      T=p*x+q*y+r*z  →   T_3 = p·T_0 + q·T_1 + r·T_2

#6:   (eq1 := 57120 = p·0 + q·48 + r·1680) = eq1 := 57120 = 48·q + 1680·r

#7:   T_5 = p·T_2 + q·T_3 + r·T_4

#8:   eq2 := 1940448 = p·48 + q·1680 + r·57120

#9:   T_6 = p·T_3 + q·T_4 + r·T_5

#10:  eq3 := 65918160 = p·1680 + q·57120 + r·1940448

#11:  SOLVE([eq1, eq2, eq3], [p, q, r]) = [p = 1 ∧ q = -35 ∧ r = 35]

      It seems that it is recursive over three terms,
      when using the first 4 terms.

#12:  T_3 = p·T_0 + q·T_1 + r·T_2

#13:  57120 = 48·q + 1680·r

     ... next triples of elements are used ...

#14:  T_7 = p·T_4 + q·T_5 + r·T_6

#15:  76069501248 = 1·1940448 + (-35)·65918160 + 35·2239277040

#16:  76069501248 = 76069501248

      It seems that it is recursive over three terms,
      when using the terms T_4 to T-7.
```

Hence, the so-called mysterious/curious series is just a further example of a **Tribonacci series**, a GAS (General Admixture Series – see details in DNL#93) to be named henceforth "The Glockenspiel Sequence".

#18: $$S\_x = \frac{0 + (48 - 35\cdot 0)\cdot x + (-1680 - 35\cdot 48 - (-35)\cdot 0)\cdot x^2}{1 - 35\cdot x - (-35)\cdot x^2 - 1\cdot x^3}$$

#19: $$S\_x = -\frac{48\cdot x\cdot(70\cdot x - 1)}{x^3 - 35\cdot x^2 + 35\cdot x - 1}$$

#20: This, above, is the General Tribonacci Generating Function.

#22: $$\mathrm{TAYLOR}\left(-\frac{48\cdot x\cdot(70\cdot x - 1)}{x^3 - 35\cdot x^2 + 35\cdot x - 1}, x, 0, 6\right)$$

#23: $$2374994160\cdot x^6 + 69913200\cdot x^5 + 2057952\cdot x^4 + 60480\cdot x^3 + 1680\cdot x^2 - 48\cdot x$$

We factorise the denominator where a, b and c are the reciprocals of the roots of the cubic see page 13).

#24: $\mathrm{FACTOR}(x^3 - 35\cdot x^2 + 35\cdot x - 1, \mathrm{Radical}, x)$

#25: $(x - 1)\cdot(x + 12\cdot\sqrt{2} - 17)\cdot(x - 12\cdot\sqrt{2} - 17)$

#26: Hence $a = 1$, $b = 17 + 12\sqrt{2}$, $c = 17 - 12\sqrt{2}$

#27: $r = a + b + c$

#28: $-q = a\cdot b + b\cdot c + c\cdot a$

#29: $p = a\cdot b\cdot c$

––– Check:

#30: SOLUTIONS([a + b + c = 35, a·b + b·c + a·c = 35, a·b·c = 1], [a, b, c])

#31: $$\begin{bmatrix} 1 & 12\cdot\sqrt{2} + 17 & 17 - 12\cdot\sqrt{2} \\ 1 & 17 - 12\cdot\sqrt{2} & 12\cdot\sqrt{2} + 17 \\ 12\cdot\sqrt{2} + 17 & 1 & 17 - 12\cdot\sqrt{2} \\ 12\cdot\sqrt{2} + 17 & 17 - 12\cdot\sqrt{2} & 1 \\ 17 - 12\cdot\sqrt{2} & 1 & 12\cdot\sqrt{2} + 17 \\ 17 - 12\cdot\sqrt{2} & 12\cdot\sqrt{2} + 17 & 1 \end{bmatrix}$$

----------- Referring to DNL#93 (David Halprin, Recursive Series of Numbers):

#32: $$\mathrm{TSN1}(T0, T1, T2, a, b, c, n) := \frac{a^{n}\cdot(T0\cdot b\cdot c - T1\cdot(b + c) + T2)}{(a - b)\cdot(a - c)} + \frac{b^{n}\cdot(T0\cdot a\cdot c - T1\cdot(a + c) + T2)}{(a - b)\cdot(c - b)} + \frac{c^{n}\cdot(T0\cdot a\cdot b - T1\cdot(a + b) + T2)}{(a - c)\cdot(b - c)}$$

#33: $\mathrm{TSN1}(0, 48, 1680, 1, 12\cdot\sqrt{2} + 17, 17 - 12\cdot\sqrt{2})$

#34: $$(\sqrt{2} - 1)^{4\cdot n}\cdot\left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right) + (\sqrt{2} + 1)^{4\cdot n}\cdot\left(\frac{\sqrt{2}}{2} + \frac{3}{4}\right) - \frac{3}{2}$$

#35: This, above, is the general term for T_n

#36: $$\mathrm{VECTOR}\left((\sqrt{2} - 1)^{4\cdot n}\cdot\left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right) + (\sqrt{2} + 1)^{4\cdot n}\cdot\left(\frac{\sqrt{2}}{2} + \frac{3}{4}\right) - \frac{3}{2}, n, 0, 5\right) = [0, 48, 1680, 57120, 1940448, 65918160]$$

#37: $\mathrm{TSN1}(0, 48, 1680, 1, 12\cdot\sqrt{2} + 17, 17 - 12\cdot\sqrt{2}, 60)$

#38: 10706705582696766753047563512512351255317254875355229964943512357373~
089090913266413193434400

#39: $$T(n) := (\sqrt{2} - 1)^{4\cdot n}\cdot\left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right) + (\sqrt{2} + 1)^{4\cdot n}\cdot\left(\frac{\sqrt{2}}{2} + \frac{3}{4}\right) - \frac{3}{2}$$

#40: TABLE(T(n), n, 0, 10) =

0	0
1	48
2	1680
3	57120
4	1940448
5	65918160
6	2239277040
7	76069501248
8	2584123765440
9	87784138523760
10	2982076586042448

#41: T(10) = 2982076586042448

#42: $T(10) = 2.982076586\cdot 10^{15}$

#43: $T(10) = 2^{4}\cdot 3\cdot 19\cdot 23\cdot 29\cdot 41\cdot 59\cdot 353\cdot 5741$

#44: This, above, is T_10 and it is 16 digits long.

Following T(20), T(30), up to …

#74: $T(70) = 2.19120633\cdot 10^{107}$

#75: T(70) =

$$2^4\cdot 3\cdot 13^2\cdot 19\cdot 29\cdot 41\cdot 59\cdot 71\cdot 113\cdot 139\cdot 239\cdot 281\cdot 337\cdot 419\cdot 569\cdot 1279\cdot 70139\cdot 2159\sim$$

$$83\cdot 2118059\cdot 17934071\cdot 1800193921\cdot 2644939853\cdot 353183656631413\cdot 272115668\sim$$

$$7009413567$$

needs 1731 sec

#76: This, above, is T_70 and it is 108 digits long, making it > 10^7 x Googol.

#77: T(100) =

1878302404620317437863890267858534586362769496501117661807600847859608880688774251304245442426950034640334203651446013089922527053368022718489906125364448

#78: $T(100) = 1.878302404\cdot 10^{153}$

#79: This, above, is T_100 and it is 154 digits long, making it > 10^53 x Googol

#80: I asked Derive to attempt to factorise it directly and after 17 hours I realised that it was too ambitious to wait any longer

Finally let's compare David's formula and Sloane's formula (DNL#107):

$$a\_(n) := \frac{-6 + (3 - 2\cdot\sqrt{2})\cdot(17 + 12\cdot\sqrt{2})^{-n} + (3 + 2\cdot\sqrt{2})\cdot(17 + 12\cdot\sqrt{2})^{n}}{4}$$

$$T(n) - a\_(n) = (\sqrt{2} - 1)^{4\cdot n}\cdot\left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right) + (\sqrt{2} + 1)^{-4\cdot n}\cdot\left(\frac{\sqrt{2}}{2} - \frac{3}{4}\right)$$

$$(T(n) - a\_(n))\cdot(\sqrt{2} + 1)^{+4\cdot n} = 0$$

See David's next mail from November:

Josef

I replied to you before lunch today, with the seemingly vain hope, that I may one day solve the Glockenspiel sum. I just didn't have a great deal of confidence.

Then I had lunch and decided to take a nap to escape the early heat wave *draussen*. As I got into bed I had an idea but decided to sleep on it anyway. Then, back with Derive, and within a few minutes, voila, there it was.

It was not as I expected it to look, since I was searching for a recurring pattern, which was my assumptive error.

Far better to use the Calculus Sum command! Now I feel like a Dummkopf for not realizing it earlier.

Enjoy and Ciao, David

$$\sum_{k=0}^{n} T(k) = (\sqrt{2}-1)^{4\cdot n}\cdot\left(\frac{3}{4}-\frac{17\cdot\sqrt{2}}{32}\right)+(\sqrt{2}+1)^{4\cdot n}\cdot\left(\frac{17\cdot\sqrt{2}}{32}+\frac{3}{4}\right)-\frac{3\cdot(n+1)}{2}$$

This, above, generates the general equation for the Sum S(n).

$$S(n) := (\sqrt{2}-1)^{4\cdot n}\cdot\left(\frac{3}{4}-\frac{17\cdot\sqrt{2}}{32}\right)+(\sqrt{2}+1)^{4\cdot n}\cdot\left(\frac{17\cdot\sqrt{2}}{32}+\frac{3}{4}\right)-\frac{3\cdot(n+1)}{2}$$

TABLE(S(n), n, 0, 10)

0	0
1	48
2	1728
3	58848
4	1999296
5	67917456
6	2307194496
7	78376695744
8	2662500461184
9	90446638984944
10	3072523225027392

After having finished preparing David's paper for this DNL I wondered if possibly the quotient of consecutive sums would also tend to a limit?

So, I tried: Numerical results confirmed my suspicion. But the limit calculated by DERIVE disappointed me giving 1. L'Hopital gave also 1 … but taking the 2nd derivative I received a satisfying result. It is the same limit as taking the quotient of consecutive terms of the series!!

Josef

See below srat …

$$srat := \frac{S(n+1)}{S(n)}$$

$$srat := \frac{(\sqrt{2}-1)^{4\cdot n}\cdot(408\cdot\sqrt{2}-577)+(\sqrt{2}+1)^{4\cdot n}\cdot(408\cdot\sqrt{2}+577)-24\cdot\sqrt{2}\cdot(n+2)}{(\sqrt{2}-1)^{4\cdot n}\cdot(12\cdot\sqrt{2}-17)+(\sqrt{2}+1)^{4\cdot n}\cdot(12\cdot\sqrt{2}+17)-24\cdot\sqrt{2}\cdot(n+1)}$$

VECTOR(srat, n, 1, 501, 100)

[36, 33.97056274, 33.97056274, 33.97056274, 33.97056274, 33.97056274]

$$\lim_{n\to\infty} srat = 1$$

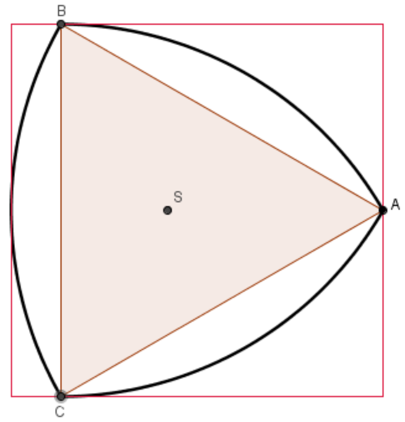
$$\lim_{n\to\infty}\frac{\text{EXPAND}\left(\left(\frac{d}{dn}\text{NUMERATOR}(srat)\right)\cdot(\sqrt{2}+1)^{4\cdot n}\right)}{\text{EXPAND}\left(\left(\frac{d}{dn}\text{DENOMINATOR}(srat)\right)\cdot(\sqrt{2}+1)^{4\cdot n}\right)} = 1$$

$$\lim_{n\to\infty}\frac{\text{EXPAND}\left(\left(\left(\frac{d}{dn}\right)^2\text{NUMERATOR}(srat)\right)\cdot(\sqrt{2}+1)^{4\cdot n}\right)}{\text{EXPAND}\left(\left(\left(\frac{d}{dn}\right)^2\text{DENOMINATOR}(srat)\right)\cdot(\sqrt{2}+1)^{4\cdot n}\right)} = 12\cdot\sqrt{2}+17$$

Das Reuleaux-Dreieck (RD) – *The Reuleaux Triangle* (*RT*)

Wolfgang Alvermann, Germany (in collaboration with J. Böhm and R. Pütter)

Das nach Franz Reuleaux benannte Bogendreieck (auch Dreibogengleichdick) bietet vielfältige Möglichkeiten für mathematische Überlegungen. In der Zeichnung haben das im RD steckende gleichseitige Dreieck und das Quadrat dieselbe Seitenlänge a.
Wälzt sich das Gleichdick im Quadrat ab, so beschreibt der Schwerpunkt des RD (=Schwerpunkt des Dreiecks) eine besondere Kurve.



The "Arc Triangle" named after Franz Reuleaux offers a couple of opportunities for mathematical considerations.
The equilateral triangle ABC and the circumscribed square have same side length a. When the orbiform curve rolls off in the square then its barycenter S (= barycenter of ∆ABC) describes a special curve (locus).

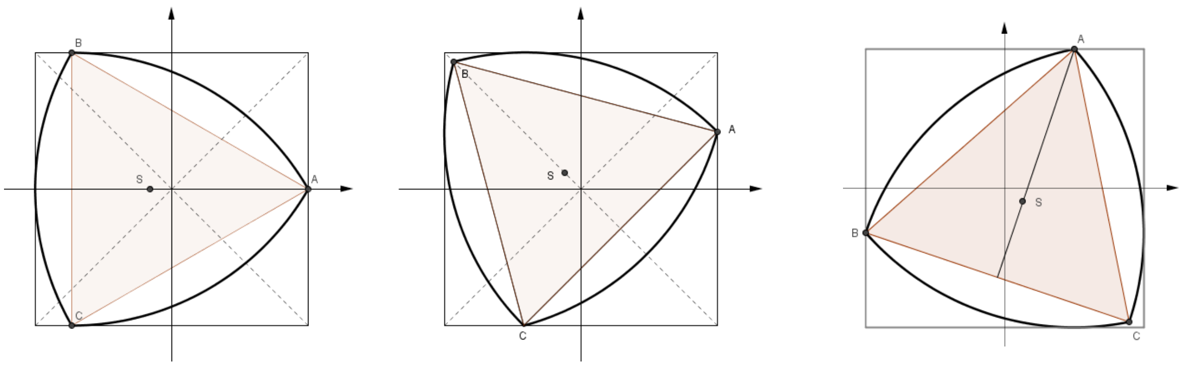
Eine einfache Animation dazu findet man in - *An animation can be found in*

https://de.wikipedia.org/wiki/Gleichdick#/media/File:Reuleaux_triangle_Animation.gif

http://mathworld.wolfram.com/ReuleauxTriangle.html

Um die Schwerpunktsbahn zu ermitteln, werden drei Ausgangspositionen des RD im Quadrat betrachtet und jeweils der Schwerpunkt S in Abhängigkeit von a ermittelt. Auf diese Weise erhält man 12 Punkte der zu bestimmenden Kurve, da aus jedem Schwerpunkt durch Achsenspiegelung bzw. durch Drehung um 90°, 180° und 270° um den Koordinatenursprung jeweils drei weitere Schwerpunkte erzeugt werden können.

Three positions of the RT are observed and the coordinates of the barycenter dependent on a are calculated. These points can be reflected of the axes which gives totally 12 points of the requested curve.



Links ist $A = \left(\frac{a}{2};0\right)$, im mittleren Bild soll B auf der 2. Winkelhalbierenden liegen, A und C parallel zur ersten Winkelhalbierenden und im dritten Bild hat A die Koordinaten $A = \left(\frac{a}{4};\frac{a}{2}\right)$.

The first position is with $A = \left(\frac{a}{2};0\right)$*, in the second position lies B on the 2nd median (AC is parallel to the 1st median) and in the 3rd position is* $A = \left(\frac{a}{4};\frac{a}{2}\right)$*.*

Man erhält die folgenden Schwerpunktslagen – *This gives the following three positions of the barycenter*:

linkes Bild (*left*) $S = \left(-a\cdot\left(\frac{\sqrt{3}}{3}-\frac{1}{2}\right);0\right) \approx (-0.07735\cdot a;0)$

mittleres Bild (*center*) $S = \left(-\frac{a}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right);\frac{a}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right)\right) \approx (-0.0577\cdot a;0.0577\cdot a)$

rechtes Bild (*right*) $S = \left(a\cdot\left(\frac{\sqrt{21}}{24}-\frac{1}{8}\right);a\cdot\left(\frac{1}{2}-\frac{\sqrt{3}}{8}-\frac{\sqrt{7}}{8}\right)\right) \approx (-0.06594\cdot a;0.047225\cdot a)$

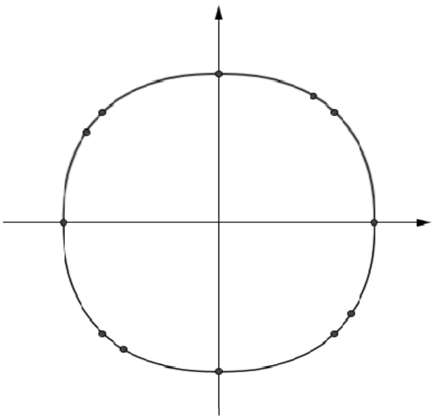
(*Josef's comment: Nice problem for students to find the coordinates of points S!*)

Trägt man diese 12 Schwerpunkte in ein KS ein, stellt man fest, dass diese nicht auf einem Kreis liegen, sondern dass

im ersten Fall $\overline{MS} = 0.07735\cdot a$,

im 2. Fall $\overline{MS} = 0.08157\cdot a$

und im 3. Fall $\overline{MS} = 0.081106\cdot a$ beträgt.



Es handelt sich vielmehr, „closely approximated by a superellipse“ (siehe Link), um eine Kurve der Form

$\left|\frac{x}{a}\right|^r + \left|\frac{y}{b}\right|^r = 1$, wobei a und b die Halbachsen darstellen.

Plotting these 12 barycenters we can observe that they are not lying on the circumference of a circle, but that

$\overline{MS} = 0.07735\cdot a$, $0.08157\cdot a$ and $0.081106\cdot a$ *for the three cases respectively. It is "closely approximated by a superellipse" (see Link below) a curve given by its implicit form*

$\left|\frac{x}{a}\right|^r + \left|\frac{y}{b}\right|^r = 1$ *with a and b as semi-axes.*

(http://mathworld.wolfram.com/ReuleauxTriangle.html)

Bevor wir diese Ortslinie berechnen, soll die Fläche des RD bestimmt werden.

Before finding this locus we would like to calculate the area of the RT.

$$A_{Triangle} + 3\cdot A_{Segment} \text{ or } 3\cdot A_{Sector} - 2\cdot A_{Triangle} = \frac{a^2}{2}\left(\pi-\sqrt{3}\right) \approx 0.70477a^2$$

Zunächst soll die Gleichung der „Superellipse“ bestimmt werden (mit TI-NspireCAS).

We will derive the equation of the approximating "superellipse" (using TI-NspireCAS).

12 points (positions of the barycenter when moving the RT)

We introduce a slider for a in the Graphs App and plot the points as a scatter diagram.

$\mathbf{sx} := \left\{ -\mathbf{a}\cdot\left(\frac{\sqrt{3}}{3}-\frac{1}{2}\right), \frac{-\mathbf{a}}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), \mathbf{a}\cdot\left(\frac{\sqrt{21}}{24}-\frac{1}{8}\right), \mathbf{a}\cdot\left(\frac{\sqrt{3}}{3}-\frac{1}{2}\right), \frac{\mathbf{a}}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), -\mathbf{a}\cdot\left(\frac{\sqrt{21}}{24}-\frac{1}{8}\right), 0, \frac{\mathbf{a}}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), \mathbf{a}\cdot\left(\frac{1}{2}\right.\right.$

$\mathbf{sy} := \left\{ 0, \frac{\mathbf{a}}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), \mathbf{a}\cdot\left(\frac{1}{2}-\frac{\sqrt{7}}{8}-\frac{\sqrt{3}}{8}\right), 0, \frac{\mathbf{a}}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), \mathbf{a}\cdot\left(\frac{1}{2}-\frac{\sqrt{7}}{8}-\frac{\sqrt{3}}{8}\right), -\mathbf{a}\cdot\left(\frac{\sqrt{3}}{3}-\frac{1}{2}\right), \frac{-\mathbf{a}}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), \mathbf{a}\cdot\left(\frac{\sqrt{21}}{24}\right.\right.$

3 points of the list (1st quadrant) – are used to derive the "Superellipse").

$\mathbf{s1} := \left\{ b\cdot\left(\frac{\sqrt{3}}{3}-\frac{1}{2}\right), 0 \right\}$

$\mathbf{s2} := \left\{ \frac{b}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right), \frac{b}{4}\cdot\left(\sqrt{2}+\frac{\sqrt{6}}{3}-2\right) \right\}$

$\mathbf{s3} := \left\{ b\cdot\left(\frac{\sqrt{21}}{24}-\frac{1}{8}\right), b\cdot\left(\frac{1}{2}-\frac{\sqrt{7}}{8}-\frac{\sqrt{3}}{8}\right) \right\}$

Superellipse se : $\mathbf{se} := \left|\frac{x}{t}\right|^r + \left|\frac{y}{t}\right|^r = 1 \;\blacktriangleright\; \left(|x|^r + |y|^r\right)\cdot |t|^{-r} = 1$

$\mathbf{se1} := \mathbf{se}|t=\mathbf{s1}[1] \text{ and } x=\mathbf{s2}[1] \text{ and } y=\mathbf{s2}[2] \;\blacktriangleright\; 2\cdot\left(\frac{\left(\sqrt{6}+3\cdot\left(\sqrt{2}-2\right)\right)\cdot\left(2\cdot\sqrt{3}+3\right)}{6}\right)^r = 1$ ⚠

Mit nSolve können wir r berechnen. Obere und untere Hälfte der Superellipse werden getrennt gezeichnet.

nSolve works to calculate the exponent r. We plot the upper and lower half of the curve.

$\text{nSolve}\left(2\cdot\left(\frac{\left(\sqrt{6}+3\cdot\left(\sqrt{2}-2\right)\right)\cdot\left(2\cdot\sqrt{3}+3\right)}{6}\right)^r = 1, r\right) \;\blacktriangleright\; 2.36185073$

So, we have both important parameters for this curve:

t = b (√ 3/3 – 1/2), r ≈ 2.361851

We solve *se* for y in order to obtain a function which can be plotted:

(because we cannot plot the implicit form of *se*.

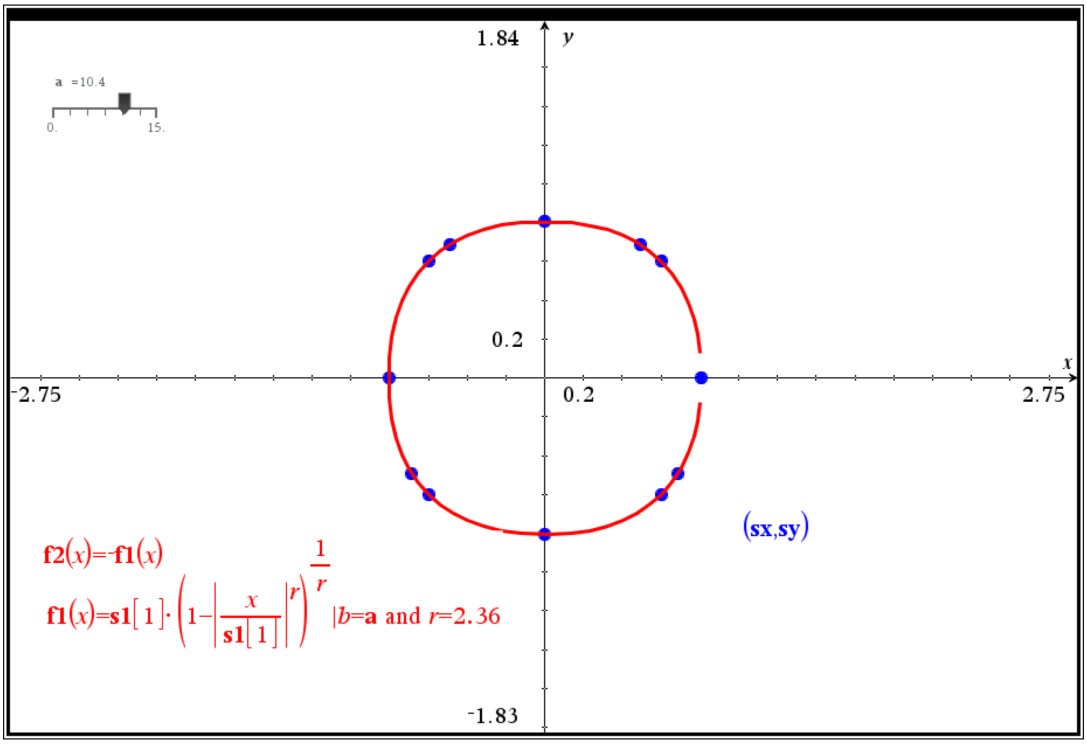
y = ± t (1 – abs(x/t)^r) ^(1/r)

Functions f1 and f2 define upper and lower half of the curve.

$\mathbf{f1}(x) := \mathbf{s1}[1]\cdot\left(1-\left|\frac{x}{\mathbf{s1}[1]}\right|^r\right)^{\frac{1}{r}} \;|b=\mathbf{a} \text{ and } r=2.361851$

$\mathbf{f2}(x) := -\mathbf{f1}(x)$

Later in this paper we will find out the exact form of the locus.



(Comment: see also DNL94/95 from 2014: Mazda sign is a superellipse …)

Die exakte Ortslinie des Mittelpunkts – *The exact locus of the center*

oder „Wie man quadratische Löcher bohrt?“ – *or „How to drill square holes?”*

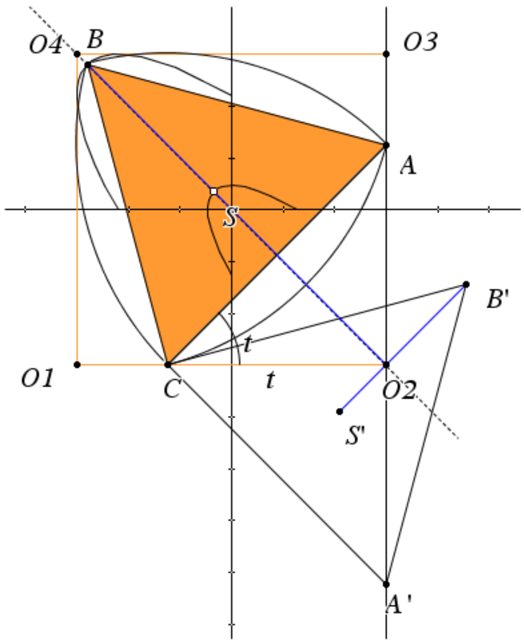
Let’s come back to Wolfgang’s paper first:

„Eine interessante Eigenschaft des Reuleaux-Dreiecks ist, dass man es als Bohrer für quadratische Löcher verwenden kann. Natürlich geht das nur mit einem Trick: Denn wie immer ein Bohrer geformt ist – wenn er sich um eine feste Achse dreht, wird er ein rundes Loch bohren. Wenn man aber ein Reuleaux-Dreieck in einer quadratischen Schablone dreht – und dabei auch der Achse eine gewisse Bewegungsfreiheit einräumt –, entsteht automatisch ein fast quadratisches Loch. An den Ecken stimmt es nicht genau, denn das Reuleaux-Dreieck bildet an seinen Ecken einen Winkel von 120 Grad. Damit kommt man beim besten Willen nicht in eine 90-Grad-Ecke. Die Idee eines quadratischen Bohrers ließ sich bereits der britische Ingenieur Harry James Watt 1914 patentieren“. (BdW 10/2009)

“It is an interesting property of the RT that it can be used to drill holes in form of a square. Of course, you must apply a trick, because which drill you ever will take it will give a circular hole – if rotated around a fix axis. But rotating a RT in a square former plate will produce automatically an – almost – square hole (if you give the axis some free moving space). It does not fit exactly in the vertices because the RT forms there an angle of 120°. So, it is impossible to reach a 90° corner. The British engineer Harry James Watt had this idea patented in 1914.” (BdW 10/2009)

There are two ways to find the locus of the vertices and of the center point as well:

- We can do it analytically calculating the locus of points C and S when segment CA with length a moves on the two lines O1O2 and O2O3. This needs a lot of calculations and manipulations. (Take t = O2C or t = ∡ACO2 as parameter. See reuleaux.dfw, reul_locus.dfw, reuleaux.tns and reuleaux_e.tns)
- Or we know from Kinematics that moving a segment on two straight lines is the so called “elliptical motion” (“Ellipsenbewegung”) where all points fixed to the segment in any way are moving along an ellipse.



The 2nd median is a symmetry axis and the two positions with AC forming an angle of 45° with O1O2 (see graph) give the vertices of the ellipses and then it is easy work to deduce the length of their axes.

(O2B and O2B’; O2S and O2S’)

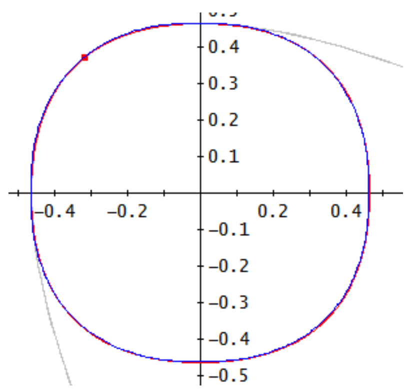
Locus of B: $ell1(t)=\begin{pmatrix} x1(t) \\ y1(t) \end{pmatrix}=\begin{pmatrix} -\frac{a}{2}\cdot\left(\cos(t)+\sqrt{3}\cdot\sin(t)-1\right) \\ \frac{a}{2}\cdot\left(\sqrt{3}\cdot\cos(t)+\sin(t)-1\right) \end{pmatrix}$ with $\frac{\pi}{6}\le t\le\frac{\pi}{3}$

Locus of S: $ell2(t)=\begin{pmatrix} x1(t) \\ y1(t) \end{pmatrix}=\begin{pmatrix} -\frac{a}{6}\cdot\left(3\cdot\cos(t)+\sqrt{3}\cdot\sin(t)-3\right) \\ \frac{a}{6}\cdot\left(\sqrt{3}\cdot\cos(t)+3\cdot\sin(t)-3\right) \end{pmatrix}$ with $\frac{\pi}{6}\le t\le\frac{\pi}{3}$

The respective lengths of major and minor axis of the ellipses are:

$\left[\frac{a}{2}\left(\sqrt{3}+1\right),\frac{a}{2}\left(\sqrt{3}-1\right)\right]$ resp. $\left[\frac{a}{6}\left(3+\sqrt{3}\right),\frac{a}{6}\left(3-\sqrt{3}\right)\right]$.

The DERIVE plot shows the exact locus (four ellipse arcs – in red -, one full ellipse – gray – and Wolfgang’s approximation by the “super ellipse” – in blue.

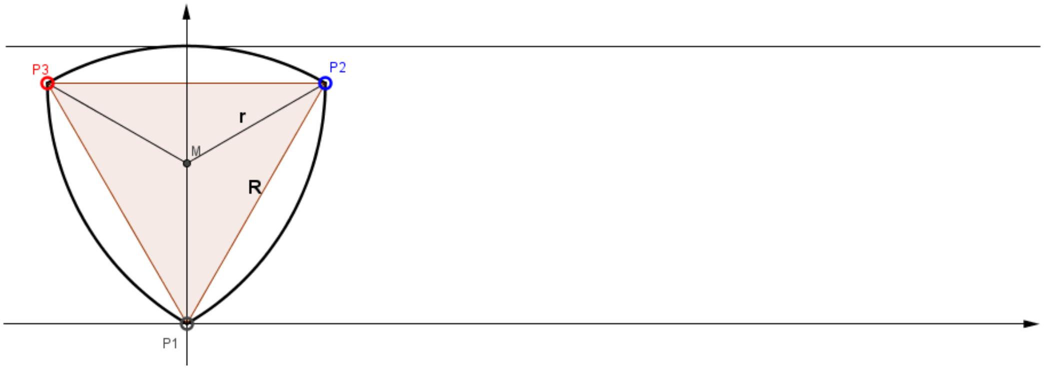


Reuleaux-Dreieck im Parallelstreifen – *Reuleaux Triangle in a Parallel Stripe*

(inspiriert durch Georg Glaesers „*Geometrie und ihre Anwendungen*“ [1])

Gegeben ist nun folgende Figur, in dem das Bogendreieck in einem Parallelstreifen umgewälzt bzw. abgerollt werden kann. Gegeben ist r, der Radius des Umkreises des Dreiecks.

Given is the following figure: the RT is rolling within a parallel stripe with r = radius of the circumcircle of the triangle.



Berechnen Sie R und h (Höhe des Dreiecks) in Abhängigkeit von r!

Auf welcher Kurve (in Abhängigkeit von r) bewegt sich der Mittelpunkt des Bogendreiecks, wenn dieses über eine Ebene rollt?

Find R and h (altitude of the triangle) dependent on r!

Which curve (dependent on r) is described by the center M of the RT, when it rolls along the stripe?

Der gesuchte Wert für R in Abhängigkeit von r kann mit Hilfe des Kosinussatzes im gleichschenkligen Dreieck mit den Schenkellängen r und dem eingeschlossenen Winkel $\alpha = 120°$ gefunden werden:

$$R^2 = r^2 + r^2 + 2 \cdot r \cdot r \cdot \cos(120°) \to R = r \cdot \sqrt{3}$$

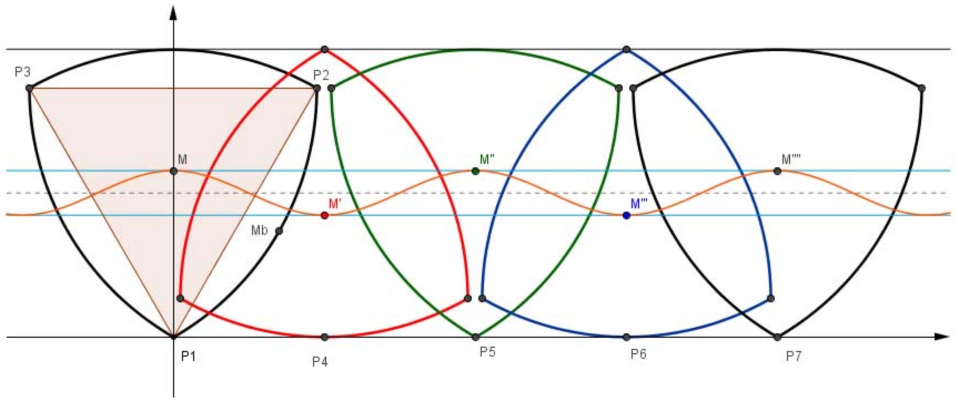
Ein gleichseitiges Dreieck mit der Seitenlänge R hat Höhe h von $h = \frac{\sqrt{3}}{2} \cdot R \to h = \frac{3}{2} \cdot r$.

We can find R applying the cosine rule in the isosceles triangle ΔMP_2P_3. Then it's easy to find its altitude h. We can do without the cosine rule, too.

r is 2/3 of h: $r = \frac{2}{3} \cdot \left(\frac{R}{2} \sqrt{3} \right) \to R = r \cdot \sqrt{3}$ *and* $r = \frac{2}{3} \cdot h \to h = \frac{3}{2} \cdot r$.

Glaeser zeigt auf Seite 296 eine ähnliche Grafik und schreibt: „Die Bahnkurven der Ecken sind teilweise geradlinig, teilweise gespitzte Radlinien. Die Bahn des Mittelpunkts ist sinusförmig.“ So nehmen wir dies zum Anlass, die Kurve durch eine Sinusschwingung zu beschreiben!

Glaeser presents a similar graph on page 296 and writes: “The loci of the vertices are partially straight lines, partially cycloids. The locus of the center is sinusoidal.” So let's take this as cause to describe the curve by a sine wave.



- In der Ausgangsstellung hat M die Koordinaten M = (0;r); dies ist der höchstmögliche Punkt.

- Rollt der Bogen $\widehat{P_1P_2}$ zur Hälfte (30°) ab, bewegt sich Mb zum Punkt $P4 = \left(\frac{\pi}{6}\cdot R;0\right) = \left(\frac{\pi\cdot\sqrt{3}}{6}\cdot r;0\right)$ (siehe Zeichnung);

  M‘ hat die Koordinaten $M' = \left(\frac{\pi}{6}\cdot R;R-r\right) = \left(\frac{\pi\cdot\sqrt{3}}{6}r;\left(\sqrt{3}-1\right)\cdot r\right)$. Dies ist der tiefst-mögliche Punkt.

- Rollt der Bogen $\widehat{P_1P_2}$ ganz (60°) ab, hat P2 die Position von P5 und M“ hat wieder den Höchstwert. $P5 = \left(\frac{\pi}{3}\cdot R;0\right) = \left(\frac{\pi\cdot\sqrt{3}}{3}\cdot r;0\right)$; $M'' = \left(\frac{\pi}{3}\cdot R;r\right) = \left(\frac{\pi\cdot\sqrt{3}}{3}r;r\right)$

- Dies setzt sich so fort; man erkennt in der Grafik eine Schwingung, die entweder als sinus- oder cosinus-Kurve beschrieben werden kann. Eine allgemeine Sinuskurve hat die Form *r* = *a* · sin(*b* · *x* – *c*) + *d* mit

  - $a = r - \frac{R}{2} = \left(2-\sqrt{3}\right)\cdot\frac{r}{2}$
  - Die Periodenlänge p beträgt $p = \frac{\pi}{3}\cdot R$ also $\frac{1}{6}$ der Periodenlänge eines Kreises mit dem Radius R $\rightarrow b = \frac{6}{R} = \frac{2\cdot\sqrt{3}}{r}$.
  - Die Phasenverschiebung c ergibt sich zu $c = \frac{\pi}{2}$.
  - Die Phasenlage d hat den Wert $d = \frac{R}{2} = \frac{\sqrt{3}}{2}\cdot r$.

Damit bewegt sich der Mittelpunkt M auf der Bahn

$$f(x,r) = -\left(2-\sqrt{3}\right)\cdot\frac{r}{2}\cdot\sin\left(\frac{2\cdot\sqrt{3}}{r}\cdot x-\frac{\pi}{2}\right)+\frac{\sqrt{3}}{2}\cdot r =$$

$$= \left(2-\sqrt{3}\right)\cdot\frac{r}{2}\cdot\cos\left(\frac{2\cdot\sqrt{3}}{r}\cdot x\right)+\frac{\sqrt{3}}{2}\cdot r$$

- *The initial position of M is M = (0;r); this is the highest possible point.*
- *Rolling off the arc* $\widehat{P_1P_2}$ *about its half (30°), then Mb moves to* $P4=\left(\frac{\pi}{6}\cdot R;0\right)=$

  $=\left(\frac{\pi\cdot\sqrt{3}}{6}\cdot r;0\right)$ *(see the sketch above);*

  The coordinates of M' are $M'=\left(\frac{\pi}{6}\cdot R;R-r\right)=\left(\frac{\pi\cdot\sqrt{3}}{6}r;\left(\sqrt{3}-1\right)\cdot r\right)$ *This is the lower most point.*
- *Rolling the whole arc* $\widehat{P_1P_2}$ *(60°) ab, then P2 is in position P5 and M" reaches again the maximum value.* $P5=\left(\frac{\pi}{3}\cdot R;0\right)=\left(\frac{\pi\cdot\sqrt{3}}{3}\cdot r;0\right)$; $M''=\left(\frac{\pi}{3}\cdot R;r\right)=\left(\frac{\pi\cdot\sqrt{3}}{3}r;r\right)$.
- *We can continue and we observe an oscillation which can be described as sine or cosine wave. The generalized sine wave is defined as* $r=a\cdot\sin(b\cdot x-c)+d$ *with*
  - $a=r-\frac{R}{2}=\left(2-\sqrt{3}\right)\cdot\frac{r}{2}$
  - Period length p is $p=\frac{\pi}{3}\cdot R$ i.e. $\frac{1}{6}$ of the period length of a circle with radius R $\rightarrow b=\frac{6}{R}=\frac{2\cdot\sqrt{3}}{r}$.
  - Phase shift c results as $c=\frac{\pi}{2}$.
  - Phase position (vertical shift) d is given by $d=\frac{R}{2}=\frac{\sqrt{3}}{2}\cdot r$.

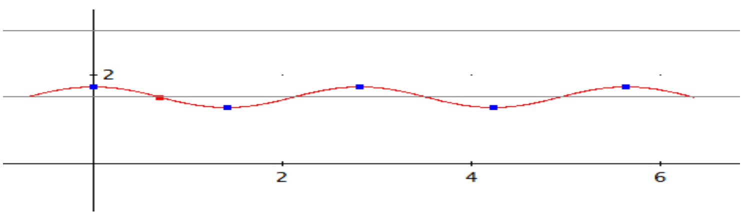
Taking this in account point center M moves along the curve:

$$f(x,r)=-\left(2-\sqrt{3}\right)\cdot\frac{r}{2}\cdot\sin\left(\frac{2\cdot\sqrt{3}}{r}\cdot x-\frac{\pi}{2}\right)+\frac{\sqrt{3}}{2}\cdot r=$$

$$=\left(2-\sqrt{3}\right)\cdot\frac{r}{2}\cdot\cos\left(\frac{2\cdot\sqrt{3}}{r}\cdot x\right)+\frac{\sqrt{3}}{2}\cdot r=\frac{R}{2\sqrt{3}}\left(2-\sqrt{3}\right)\cdot\cos\left(\frac{6}{x}\right)+\frac{R}{2}$$

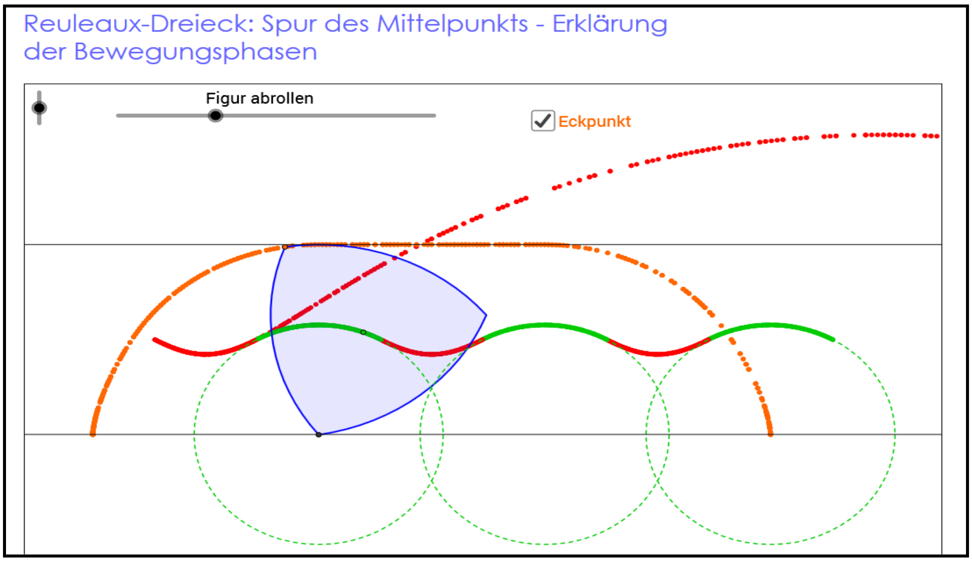
Now it was my (Josef's) turn to join the party. I had the idea to find an analytical way to confirm Wolfgang's sine wave locus or – much better – to produce an animation on the TI-Nspire (or with DERIVE using sliders). Unfortunately, I was not successful and I had my concerns because I remembered that the locus of a point fixed on a circle rolling on a straight line is a cycloid. Could it be that Glaeser does not mean a sine wave but only a graph which is similar to a sine wave?

I wrote to Benno Grabinger, Thomas Müller and Andreas Lindner. They all had a cycloid in mind. So, I tried to compose the oscillation with parts of a cycloid (DERIVE):



This looked quite nice, but is this really the right locus?

Wolfgang and I did some research in the web. We both found a great GeoGebra-animation. (https://www.geogebra.org/m/Ga0sOO1b)



Glaeser: *"The loci of the vertices are partially straight lines, partially cycloids. The locus of the center is sinusoidal."* The animation confirms Glaeser's comment.

The GeoGebra file does neither unveil the equation of the cycloid arcs nor of the locus in question.. It traces the center point.

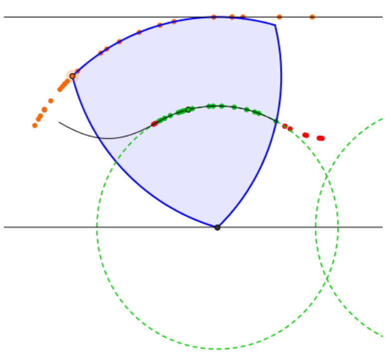
I made a model of the RD of paper board and rolled it off between two straight edges. I found the following functions which are forming partially the locus of the center:

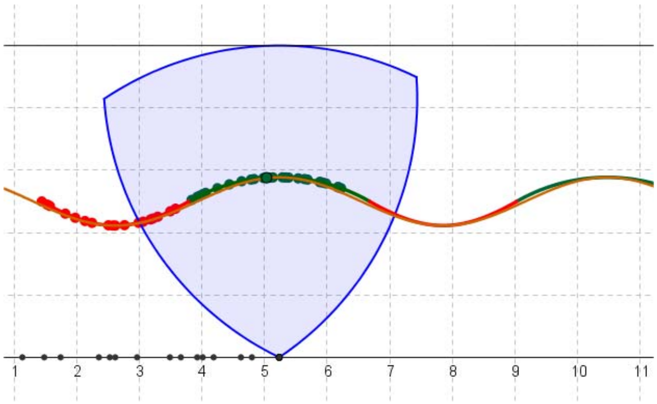
The green coloured parts are arcs of a circle:

$$\left(\frac{R}{\sqrt{3}}\cos(t)+k\cdot\frac{R\pi}{3};\frac{R}{\sqrt{3}}\sin(t)\right),\ \frac{\pi}{3}\le t\le\frac{2\pi}{3},\ k=1,2,3,...$$

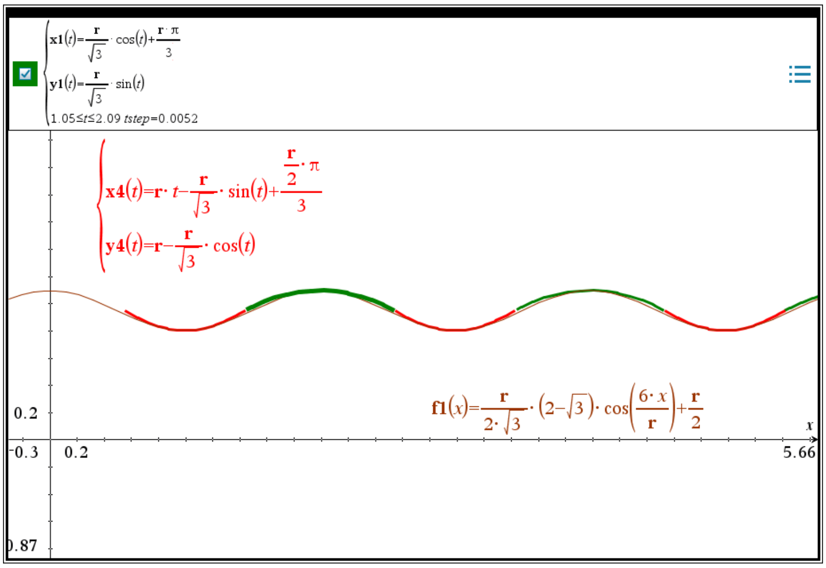
The red coloured parts are arcs of a prolate cycloid:

$$\left(R\cdot t-\frac{R}{\sqrt{3}}\sin(t)+(2k-1)\frac{R}{2}\cdot\frac{\pi}{3};R-\frac{R}{\sqrt{3}}\cos(t)\right),\ -\frac{\pi}{6}\le t\le\frac{\pi}{6},\ k=1,2,3,...$$





The brown curve is the approximating sine wave (in its cosine form).

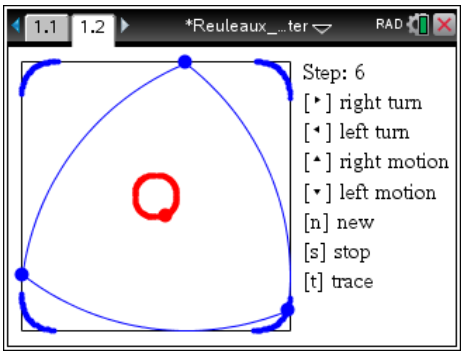


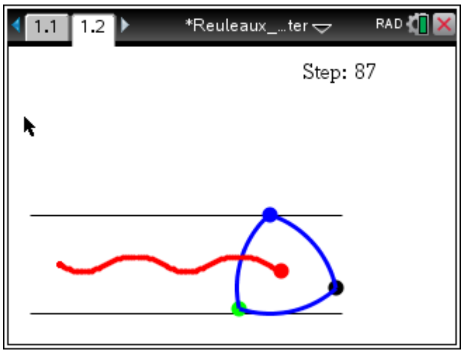
https://www.geogebra.org/m/aZa7Dknw

https://www.geogebra.org/material/show/id/WJej2erP

Das große TI-Nspire-Finale – *The great TI-Nspire-Final*

When I tried to find some related Nspire or DERIVE materials I came across Rolf Pütter's great collection of TI-Nspire-LUA files. And there was one Reuleaux-Animation among them. I wrote a mail and asked it it would be possible to add the parallel stripe RT, too. To my great surprise I received an updated version some days later. Many thanks to Rolf Pütter. See the link to his wonderful website on our Information page and in the references.





Reuleaux Triangle, version 1.1, 2018-01-31, Rolf Pütter (see References)

A closed convex curve in the plane is placed between two parallel lines, which are moved towards each other until they touch the curve. The distance between the touching parallels is then called the "width" of the curve in the direction normal to the parallels.
The circle has the same width for all directions. Are there other curves with this property? Yes, there are. The simplest of them is the Reuleaux triangle. It is formed connecting the vertices of an equilateral triangle by three circular arcs of 60 ° each, with center at the third vertex and radius equal to the edge length of the triangle. The width of a Reuleaux triangle is the same as this radius.
A Reuleaux triangle may be rotated freely in a square with edge length equal to its width. The program shows this motion and the motion without slip on a straight line.
Via Menu>Movement you can switch between two modes: Square and Straight line. The key commands are the same in both modes.

Comments on the animation files:

Rolf Pütter's animation: The LUA-script consists of more than 340 lines of code!!
The GeoGebra files shows also 340 steps in its construction protocol.
So, it seems to be obvious that this is not so easy done, Josef.

I collect all references:

Referenzen – *References*

[1] Georg Glaser, Geometrie und ihre Anwendungen in Kunst, Natur und Technik Elsevier GmbH München, 2005S.296-298

Internet:

https://de.wikipedia.org/wiki/Gleichdick#/media/File:Reuleaux_triangle_Animation.gif

http://didaktik.mathematik.hu-berlin.de/files/reuleuaux_kostrzewski_belegarbeit.pdf

http://mathworld.wolfram.com/ReuleauxTriangle.html

http://www.mikesenese.com/DOIT/2011/10/drilling-square-holes-with-a-reuleaux-triangle/

https://www.geogebra.org/m/P4wB9nXs

https://www.geogebra.org/m/aZa7Dknw

https://www.geogebra.org/m/sjCbkKW3

https://www.geogebra.org/m/EAbUMsGJ

https://www.geogebra.org/m/q4Q3KUrc

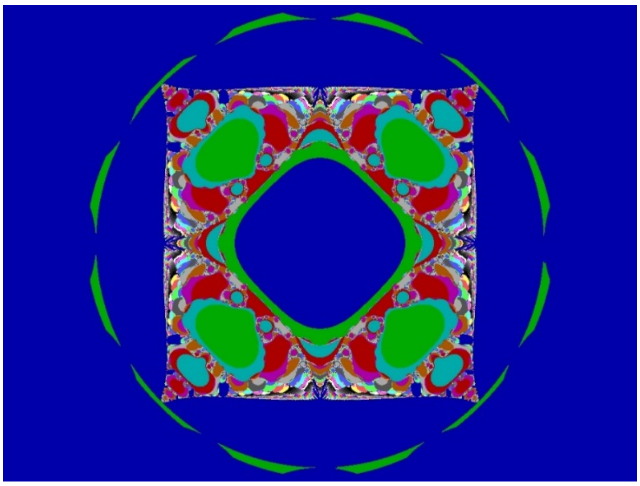
http://www.mathematische-basteleien.de/gleichdick.htm

http://www.ticalc.org/pub/nspire/lua/math/reuleaux.zip

Rolf Pütter's great collection of LUA-programs for TI-Nspire
http://www.ticalc.org/archives/files/authors/114/11430.html

https://www.ics.uci.edu/~eppstein/junkyard/reuleaux.html

http://kmoddl.library.cornell.edu/math/2/



Fractal provided by David Halprin

Mail from Leon Magiera, Poland

Hello Josef,

I have the following question: How to calculate the following integral (with a > 0)?

$$\int_0^{\infty} \frac{x^2}{(\text{COSH}(x) + 1)\cdot(\text{COSH}(x) + a)}\, dx$$

Leon

Any advice would be highly appreciated, Josef.

Communication with David Dyer

Sir:

I hope this email address remains to be active.

I am a math Ph. D. from the University of Maryland (UMCP) from the 70's. I also, belong to an unaffiliated math study group which consists of n < 4 participants.

On rare occasion, I have found DERIVE to be very useful to use DERIVE (because of my past knowledge). I am usually able to recall the different capabilities and the associated SYNTAX but I have run into brick wall with the most elementary question:

"How to I insert 'comment' lines in the center of a work-space.

For a trivial example, I might want to have the 'comment'

```
"y-prime"
```

above a sequence lines

```
y:=x^3 + exp(x^1000)
"diff(y,x)"
```

to which the output would be

```
3x^2 + x^999*exp(x^1000)
```

Obviously, this is a sequence which would be useful in a Calculus class which is struggling with the issue of when to/to not apply the basic chain rule.

I am going on......sorry.

Do you recall the necessary syntax to "tell" DERIVE" that the line "y-prime' is what we called a comment line (COM) waaaay back in the days of inputting as pile of cards for a FORTRAN program?

I sincerely hope your memory is better than mine.

Such capability would be useful while I compute a messy derivative or anti-derivative.

Many thanks in advance.

(Dr.) Dave Dyer

BTW You might be wondering why I don't just go to the help file but my "computer guy" has been unable to load any DERIVE help file (any suggestions?). I entitled to the file because I actually PAID for DERIVE back in the 1970"s.

Dear DERIVE User,

the way you edited your example is quite right.
I don't know which DERIVE version you are using, which operating system?
There are patches which make the Help-file readable.

In the latest versions you can take a text box and insert your comment(s) – as many lines as you want. In older versions you can insert your comments in quotes as you did in your mail.

This is what I authored in Derive. (I attach the respective mth-file, which should run in all versions).

```
"y-prime"

y:=x^3+EXP(x^1000)

"diff(y,x)"

DIF(y,x)

;Simp(#4)
1000*x^999*#e^(x^1000)+3*x^2
```

Best regards and thanks for your request. Much success in your n < 4 Math Group.

Josef

Thank you very kindly. All you had to say was "...." which had come back to me. However, I did discover the text box feature you described. Our group of 'old timers is having a lot of fun doing math these days. I already mentioned our common education, hence we just fit together as old friends having, in many cases, the same professor providing support for relevant difficulties. What is/were your special interests as a mathematician or Computer Scientist? Do you mind if I keep your email? I.E. May I ask you questions in the future. In an effort to familiarize yourself with what we have studied in group (n≤ 3) so far. I will ***eventually*** send you a list of the texts we've completed. (I would like to say devoured but it was a civil meal.)

Thanks again for your very specialized help.

(Dr.) Dave Dyer

Hi David,
nice communication. I am glad that I could help your memory.
It would be great if you and your math-company would share one or the other of your texts.
I recommend to join the DERIVE User Group (see link below). Membership is not connected with any duties. It is just to join a worldwide group of CAS (favorable DERIVE and TI-CAS-Calculators) enthusiasts.

About my interests: I used to be a schoolteacher (secondary and vocational school) and was very busy in pre- and in-service training of math teachers.
I retired in 2001.
Best regards to you and your group,
Josef

Josef,

Perhaps something has been lost in the translation! I have NO "COMPANY" what I am is a retired math Ph. D. who along with friends and classmates from the 70's who also want to escape "retirement lethargy" by challenging ourselves to work through advanced math reading material or fairly current textbooks. We speak on the phone, meet when practicable (the closest distance between any two of us is a boring 4 hour drive). Share ideas for solutions to Exercises. Our short list of completed items is:

(1) Galois Theory, Ian Stewart
(2) Elliptic Tales, Avner Ash & Robert Gross
(3) The Students' Guide to Maxwell's Equations, Fleush
(4) Algebraic Number Theory & Fermat's Last Theorem, Ian Sewart and David Tall
(5) An Introduction to Differential Geometry and Relativity, Robert Faber
(6) Matrix Groups for Undergraduates, Kris Tapp

contenders for our next challenge upon completion of Tapp's 2nd ed. are:
Introduction to Manifolds, Loring Tu;
Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (Studies in Non linearity), Strogatz, Steven H;
A Brief Course in Algebraic Topology, W.S. Massey and
Differential Geometry of Curves and Surfaces, Kris Tapp.

We are quite proud of our work, yet we understand the limitations to growing older (median age 66 years). We work very slowly as you may imagine, after all we are no longer grad students on top of our game.

I have attached a copy of my completed membership application. I hope it is legible and printable. Let me know if you need me to put one in the mails (par avion, n'est pas?)

I have searched older editions of the journal and can't find my Bisection Method with DERIVE which was included in an early edition, I believe, in the late 90's. Is there a search by Author feature?

I am very happy to have reached out to you and rediscovered the group.

Best Wishes for the Upcoming Holidays,

David

From DNL to Don:

Dear David,

many thanks for joining our group.
Sorry for my weak English. I had no COMPANY in commercial sense in mind.
I meant it in the sense of group, community, …

Your contribution on the Bisection Method appeared in DNL#11 page 15 (1993)
You can download the issue in its revised version from our website.

Best regards
Josef

From David to DNL:

Josef,
I absolutely love the enhancements you made to my rather simple article. They are great. However, you should understand that this was not written for advanced or even pre-calculus students.

The Intermediate Value Theorem is very suitable for my mainstream class. No one actually asked for a proof. If someone did, I invited them for a discussion in my office.
The course in question was, at the time, the basic 3-credit hurdle that all liberal arts students not in a more advanced sequence, were required to take. Leave it to say these students were crippled with "math anxieties", some of them were repeating the course for occasionally the third time. They were quietly directed to my class because I satisfied all course requirements of the course syllabus but I did so by empowering students with a CAS (DERIVE), an easy table generating program *which actually had graphics capabilities less challenging than DRIVE.

So, I need to emphasize to readers that although I could have introduced ITERATES but it not only would have been lost on them, it might have terrified some of them.

All students were pretested to determine their learning styles and they fell overwhelmingly in a category called "basic". It was the style which did not include advanced notions. EVERYTHING was done on a very basic level. The notion of "function" was about the most advanced integrated topic.

So, I appreciate your enhancements to my article but I dare say, using them would not exactly serve "basic learners"

In 1993, DERIVE was still an adolescent in the CAS world and it didn't take long for it to become a "big boy" which would not have served my study students at all.

There is a danger to integrating every new shiny object that comes with updated versions of DERIVE. These basic, often scorned, students are still here and they need to take a course which makes more advanced concepts available to them on a basic level yet still satisfies syllabus requirements. This was the course no one wanted to teach despite 30+ classes fully registered every semester. We had 30+ full time faculty, do you think everyone volunteered to take a section? No, they were sent to Dr. Dyer who taught three sections (and filled his schedule with Calculus III and Differential Equations, using advanced features of DERIVE at times). I bask in the sunlight when former students return to tell me how much they appreciated their math course. Some were afraid to change to a curriculum which included the so called "soft calculus" for Business and Social Science students and I assured them that they would be fine and to come see me if they had questions. Remarkably, only a few needed help. Most said they were just fine!

David

Mail from Don Speray, USA

Hi Josef,

I am a collector of calculating devices, from mechanical calculators (e.g., Curtas) to electronic calculators (scientific, programmable, graphing, CAS). I bought a TI-92 Plus in 1999, but never explored its CAS capabilities. Now, I have HP Primes, Casio ClassPads, and TI-Nspires. I also have a Mathematica license, and I have a new interest in CASes. I am now retired, and have lots of time.

Recently, I have developed ways to generate formulas for the sum of the first N integers to an arbitrary power, using a CAS. I have not done a literature search on this topic because I don't want to contaminate my thinking. I want to (re)discover the work by myself. So, I do not know if my work is original, but it is fun. I am writing a paper on my methods, and here is a result – the polynomial for the sum of the first N integers to the 100th power (generated by a short program for the TI-Nspire):

Don

DNL:

What followed was a huge expression which I don't want to reprint (2 pages long). But I was able to copy and paste this "monster" to DERIVE and made a try:

```
((n*(n+1)*(2*n+1)*(36465*n^(98)+1786785*n^(97)+27992965*n^(96)-
42882840*n^(95)-4912467560*n^(94)+ …
… ))/(7365930))
```

```
f(n) :=

                                   98            97               96              ~
  n·(n + 1)·(2·n + 1)·(36465·n    + 1786785·n    + 27992965·n    - 42882840·~
 ─────────────────────────────────────────────────────────────────────────~
                                                                                  ~

 95                 94                 93                     92                  ~
n   - 4912467560·n    + 7390142760·n    + 1091693643040·n    - 164123553594~
────────────────────────────────────────────────────────────────────────────~
                                                                                  ~
```

$f(2) = 1267650600228229401496703205377$

$(1 + 2)^{100} = 515377520732011331036461129765621272702107522001$

I was disappointed because the two results did not match. Maybe that I misunderstood Don's explanation. I wrote:

From DNL to Don:

Dear Don,
thanks for your mail from 6 January.
Sorry for my delay in responding. We just came back from a travel through Myanmar.
Now I am trying to finalize DNL#108 and I checked my DUG-related emails.
This is a really huge formula which you sent in your mail. Did you develop it with TI-Nspire?
I copied the expression as f(n) to DERIVE – it worked.
Then I simplified f(1) = 1 – ok and then f(2). I understand your description that this should evaluate to (1+2)^100??
But 3^100 does not equal f(2). Do I misunderstand the formula?
Can you please send the respective TI-Nspire file. I'd like to include your remarkable result in the recent DNL.
Best regards and many thanks
Josef

From Don to DNL:

Hi Josef,

It is for the sum 1^100 + 2^100 + 3^100 + … + n^100. I have recently pushed the limit to the 110-th power, and it took three days on a PC (using the Nspire Student Software emulator).

It would be nice if this is an original approach, and I am still writing a paper on it (slowly). I have three other methods to derive these formulas, but they require more Nspire resources, and they can only go up to about the 50-th power (they have to solve large linear systems). I will eventually program them in Mathematica.

Here is the Nspire code. Use like rsop(30, 47) for the sum of the 30-th powers up to n=47. Note that rsop(1,100) gives the correct answer of 1+2+3+…+100 = 5050.#

Don

```
rsop(1,100)
                         n·(n+1)
                         -------
                            2
                          5050
                          5050
                                         Fertig
rsop(100,2)
 n·(72930·n^100+3682965·n^99+61382750·n^98-9…
 ---------------------------------------------
 1267650600228229401496703205377
 1267650600228229401496703205377
                                         Fertig
```

```
"rsop" erfolg. gespeichert
Define LibPub rsop(pp,nn)=
Prgm
© Recursive sum of powers (final_p, sum_limit)
Local p,k,s,sums
DelVar n
sums:={}
For p,1,pp
s:=nCr(p+1,p+1)·n^(p+1)+nCr(p+1,p)·n^p−n
For k,1,p−1
s:=s+nCr(p+1,k)·(n^k−sums[k])
EndFor
s:=s/nCr(p+1,p):sums:=augment(sums,{s})
EndFor
© Final 's' is what we're after.
Disp s
Disp s|n=nn
Disp Σ(k^pp, k, 1, nn)   © for verification
EndPrgm
```

The TI-Nspire-output does not present the factorized expression.
Let's verify with DERIVE:

$$1^{100} + 2^{100} = 1267650600228229401496703205377$$

This works, sorry for my misunderstanding, Josef.

Don sent a second approach and announced an explanation of his methods for the next DNL.
Thanks in advance, Josef

TI-Innovator™ Hub

Plug-and-play and ready-to-use with TI graphing calculators, the TI-Innovator™ Hub enables students to learn basic coding and design, use those skills to program and build working solutions, and connect STEM concepts.

Multiple input and output ports expand the TI-Innovator™ Hub capability to motivate students to imagine, design, build and test creative solutions.

From:

https://education.ti.com/en/products/micro-controller/ti-innovator

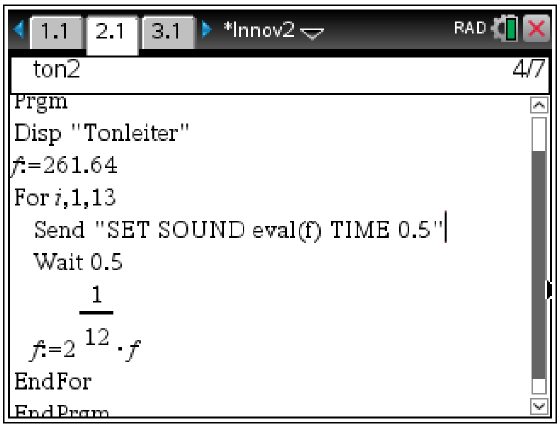
https://education.ti.com/de/products/micro-controller/ti-innovator (deutsch)

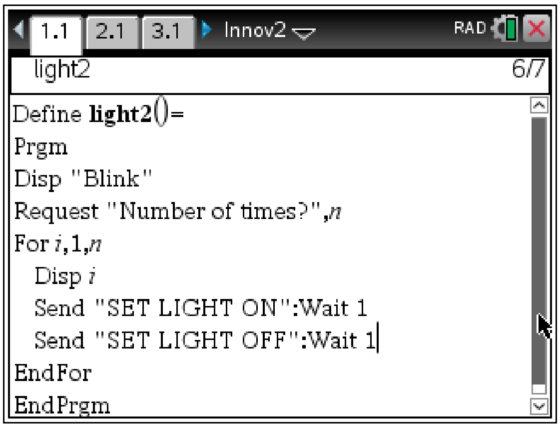
You can find an introduction in programming the Innovator in English and in German as well. 10 Minutes Coding for TI-Innovator on the following websites:

https://education.ti.com/en/activities/ti-codes/nspire/10-minutes-innovator

https://education.ti.com/de/activities/ti-codes/nspire/10-minutes-innovator

One example in German (producing a gamut) and one in English (producing blinking lights a certain number of times).



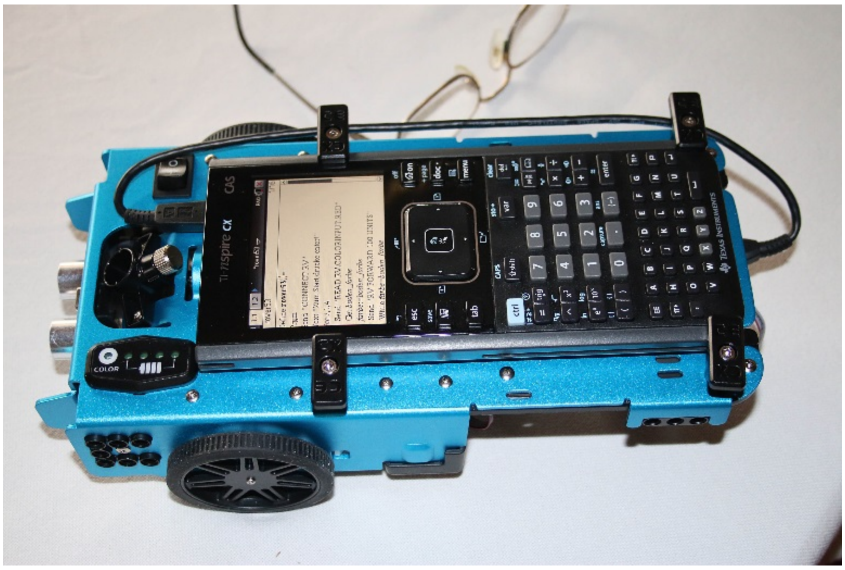


TI-Innovator™ Rover

The Hub can control the Rover which is a small vehicle with a lot of remarkable properties. Via TI-Nspire we can program this machine via the Hub.

https://education.ti.com/en/products/micro-controller/ti-innovator-rover

https://education.ti.com/en/products/micro-controller/ti-innovator-rover



Units 4 and 5 of the 10 Minutes Coding for the Innovator are devoted to the Rover. Until now only the English version for the Rover is on the web. But ist should only be a question of days that the German version is also available.

The picture above shows the rover exploring the color of the tabel cloth (in RGB values) in our living room. The rover is equipped with a couple of sensors: a color sensor, a distance sensor an LED display, a holder for a marker to draw paths on paper and a gyroscope to measure headings.

See a very short introductory program which moves the little blue vehicle along a square. But you can do much much more. The Rover will move on the floor without hitting any piece of your furniture – if programmed correctly.